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A Sahlqvist theorem for distributive modal logic

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Abstract

In this paper we consider distributive modal logic, a setting in which we may add modalities, such as classical types of modalities as well as weak forms of negation, to the fragment of classical propositional logic given by conjunction, disjunction, true, and false. For these logics we define both algebraic semantics, in the form of distributive modal algebras, and relational semantics, in the form of ordered Kripke structures. The main contributions of this paper lie in extending the notion of Sahlqvist axioms to our generalized setting and proving both a correspondence and a canonicity result for distributive modal logics axiomatized by Sahlqvist axioms. Our proof of the correspondence result relies on a reduction to the classical case, but our canonicity proof departs from the traditional style and uses the newly extended algebraic theory of canonical extensions.

Keywords: Distributive modal logic; Distributive modal algebras; Lattice expansions; Canonical extensions; Correspondence theory; Canonical logic; Canonical varieties; Duality

1. Introduction

In the theory of classical modal logic, an important role is played by what we rather loosely call *Sahlqvist theory*; with this we understand the theory that seeks to identify

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large classes of formulas that are both *canonical* and *correspond* to an elementary frame condition which can be effectively obtained from the formula. The value of such a unifying mathematical theory lies in its applications to for instance completeness results for individual logics, but also in that it deepens our understanding of the associated algebraic duality theory. When it comes to modal logics that are based on a weaker than classical propositional logic (or, algebraically, to classes of algebras based on not necessarily Boolean lattices), general canonicity and correspondence results could be equally useful, but here it seems to be only fairly recent that theory has started to take shape in its most general form. The aim of the research that we report on in this article was to help in filling this gap.

Basically, what we have done is to extend the Sahlqvist canonicity and correspondence results [25] to the more general setting of *Distributive Modal Logic* (or DML) in which we may add modalities (such as weak forms of negation) to the fragment of classical propositional logic given by conjunction, disjunction, true, and false. Thus our formulas are given as follows:

$$\phi \ ::= \ p \mid \bot \mid \top \mid \phi \wedge \phi \mid \phi \vee \phi \mid \Diamond\phi \mid \Box\phi \mid \rhd\phi \mid \lhd\phi.$$

Here the modalities $\Diamond$ and $\Box$ are meant to be disjunction and conjunction preserving modalities, respectively, whereas $\rhd$ turns disjunctions into conjunctions, and vice versa, $\lhd$ turns conjunctions into disjunctions. Thus the latter modalities can be seen as weak forms of negation, or as combinations of a classical modality with negation. (Obviously, our results also apply to settings with various modalities of each kind; the above restricted syntax in which there is exactly one operator of each kind is just to keep our notation simple and uniform.) Note that in this non-classical setting tautologies are not sufficient, so we must consider sequents to capture the logics; since we are not interested in proof theory here, we will simply take pairs of formulas, written as $\alpha \Rightarrow \beta$, as our sequents. We can then formally define a *distributive modal logic* or DML to be any set of sequents which contains certain axioms and is closed under certain natural inference rules.

Our setting is thus closely related to that of *Positive Modal Logic*, or PML, introduced by Dunn in [11] and studied further by Celani and Jansana in [6,7]. In many aspects our setting in fact *extends* that of PML, since the language of the latter does not allow the order reversing modalities $\rhd$ and $\lhd$; in this sense, many results on positive modal logic are covered by our work. On the other hand, researchers in PML have focused their attention on special interpretations in which the two modalities, $\Box$ and $\Diamond$, are closely related, whereas we make no such assumptions beforehand. For a more detailed discussion of the connection between our work and that on positive modal logic, the reader is referred to Section 6.

As usual, there are two natural ways to study distributive modal logics by semantic means: a relational and an algebraic one. Starting with the algebraic semantics, we introduce the notion of a *distributive modal algebra* or DMA as a bounded distributive lattice expanded with operations $\Diamond$, $\Box$, $\rhd$ and $\lhd$ satisfying certain laws. For the relational semantics, we define the notion of a *frame* for distributive modal logic; it will come as no surprise that just as in the relational semantics for similar logics, these structures carry, besides a binary relation for each of the modalities, also an ordering relation. As we will see, the above mentioned questions on correspondence, completeness and canonicity find their natural place in the context of distributive modal logic.

It is also in this area that the main contributions of this paper lie:

- Using a new and careful analysis in terms of signed generation trees, we extend the definition of Sahlqvist terms to this setting of distributive modal logic, identifying the notion of a *Sahlqvist sequent* for distributive modal logic;
- For these Sahlqvist sequents we will prove both a correspondence and a canonicity result. Taken together, we obtain a general completeness result for distributive modal logics that are axiomatized by Sahlqvist axioms.

It is important to spend a few words on our proof method here, since it departs from the style that is most frequently employed in modal logic. The most important divergence from 'standard modal procedure' in Sahlqvist theory is that we do *not* use the correspondence result when proving canonicity, as is done in for instance [4,7,26]. In fact, we will treat correspondence and canonicity in completely separate ways. (This, and many other aspects, makes our approach closest to that of Ghilardi and Meloni [16]; we will say more about this connection in the concluding section of this paper.)

Concerning our proof that Sahlqvist sequents *correspond* to elementary properties of distributive modal frames we can be fairly brief. We could have presented a proof which is more or less analogous to the standard proof presented in for instance [4]; as it happens, however, we could manage to actually *reduce* the correspondence result to the classical case. This reduction is based on semantic ideas and is related to the Gödel translation of intuitionistic logic into the modal logic **S4**, cf. Chagrov and Zakharyaschev [8] for an overview.

When it comes to *canonicity* for Sahlqvist sequents, our proof method takes advantage of developments within the *algebraic* theory of canonical extensions of algebraic lattice expansions (that is, lattices expanded with further operations) and is a direct generalization of Jónsson's proof of canonicity of Sahlqvist sequences [20]. The algebraic theory of canonical extensions originates with the seminal paper by Jónsson and Tarski [21] on Boolean algebras with operators; through contributions by Jónsson, Gehrke and others [12–15], the theory has recently become applicable in a far wider setting than just Boolean algebras with operators.

In this approach, the canonical extension $\mathbb{A}^\sigma$ of an algebraic lattice expansion (ALE) $\mathbb{A}$ is defined abstractly rather than using duality theory. The basic idea of this definition is to start with the canonical extension $\mathbb{L}^\sigma$ of the lattice reduct $\mathbb{L}$ of $\mathbb{A}$: without going into details, let us just mention that $\mathbb{L}^\sigma$ can be defined in terms of a (modulo isomorphism) unique embedding of $\mathbb{L}$ into a complete lattice that satisfies some density and compactness conditions [14]. This lattice extension can be expanded by an additional operation f^σ for each additional operation f of the original algebra. The elegance of the method lies in the fact that properties of f (in terms of its interaction with the lattice operations of $\mathbb{A}$) naturally induce similar, *or even better*, properties of f^σ. This leads to the formulation of transparent criteria under which the operation $(\cdot)^\sigma$ is functorial, which in its turn allows the identification of large classes of ALE equations that are canonical.

In the particular context of distributive modal algebras, we will see that the canonical extension $\mathbb{A}^\sigma$ of a DMA is what we will call a *perfect* DMA; perfect DMAs can be understood as frames in algebraic disguise. Part of our proof that the validity of Sahlqvist sequents is preserved under moving to this canonical extension, stems from the more general theory that we just described (note, however, that we have tried to make the paper self-contained). But of course, we also needed to prove a number of new results: for instance, on the behaviour of certain DMA-definable operations under the operation $(\cdot)^\sigma$ and on the rewriting of Sahlqvist DMA-sequents into a certain desirable shape.

Finally, it will be obvious that our results will also have an impact on the duality theory for distributive lattice expansions. We hope to address this issue more specifically later.

Overview of paper. In the next section we will lay out the basic framework of distributive modal logic, distributive modal algebra, and their connections. In Section 3 we give our definition of what a Sahlqvist sequent is in the context of distributive modal logic, and we formulate our main results: a canonicity and a correspondence result for Sahlqvist sequents. Section 4 is devoted to the proof of the correspondence result, while in Section 5 we give a fairly detailed and self-contained proof of the canonicity result. In Section 6 we give a couple of examples, applying our correspondence theorem to positive modal logic and to a wide range of logics and algebras with weak forms of negation. The final section contains some conclusions and questions for further research.

2. Distributive modal logics and distributive modal algebras

2.1. Distributive modal logics

Syntax

In classical modal logic modalities are added to the basic connectives of classical propositional logic. Here we want to add modalities and weak forms of negation to the fragment of classical propositional logic given by conjunction, disjunction, true, and false. Thus our connectives are $\vee, \wedge, \bot, \top, \Diamond, \Box, \rhd, \lhd$, where $\vee$ and $\wedge$ are binary, $\bot$ and $\top$ are nullary (constant), and the others are unary.

Given a set $X = \{x, y, x_1, x_2, \ldots\}$ of (propositional) variables, we may now form terms or formulas using these connectives.

In this non-classical setting tautologies are not sufficient and we must consider sequents, or pairs of formulas, to capture the logics: a *(modal) sequent* is simply a pair of distributive modal formulas. Such a pair (α, β) will be written $\alpha \Rightarrow \beta$.

Logics

The following formal system based on modal sequents is modified from Dunn [11] where a modal sequent $\alpha \Rightarrow \beta$ is called a consequence pair and is denoted $\alpha \vdash \beta$.

Definition 2.1. A *distributive modal logic* (DML) is a set Λ of modal sequents such that

(Axioms) Λ contains the following sequents:

$$x \Rightarrow x$$
$$\bot \Rightarrow x \qquad x \Rightarrow \top$$
$$x \wedge (y \vee z) \Rightarrow (x \wedge y) \vee (x \wedge z)$$
$$x \Rightarrow x \vee y \qquad y \Rightarrow x \vee y \qquad x \wedge y \Rightarrow x \qquad x \wedge y \Rightarrow y$$
$$\Diamond(x \vee y) \Rightarrow \Diamond x \vee \Diamond y \qquad \Diamond\bot \Rightarrow \bot$$
$$\Box x \wedge \Box y \Rightarrow \Box(x \wedge y) \qquad \top \Rightarrow \Box\top$$

$$\rhd x \wedge \rhd y \Rightarrow \rhd(x \vee y) \qquad \top \Rightarrow \rhd\bot$$
$$\lhd(x \wedge y) \Rightarrow \lhd x \vee \lhd y \qquad \lhd\top \Rightarrow \bot$$

(Inference rules) Λ is closed under the following inference rules:

$$\frac{\alpha \Rightarrow \beta \quad \beta \Rightarrow \gamma}{\alpha \Rightarrow \gamma}\text{ (cut)} \qquad \frac{\alpha \Rightarrow \beta}{\alpha(\gamma/x) \Rightarrow \beta(\gamma/x)}\text{ (substitution)}$$
$$\frac{\alpha \Rightarrow \gamma \quad \beta \Rightarrow \gamma}{\alpha \vee \beta \Rightarrow \gamma} \qquad \frac{\gamma \Rightarrow \alpha \quad \gamma \Rightarrow \beta}{\gamma \Rightarrow \alpha \wedge \beta}$$
$$\frac{\alpha \Rightarrow \beta}{\Diamond\alpha \Rightarrow \Diamond\beta} \qquad \frac{\alpha \Rightarrow \beta}{\Box\alpha \Rightarrow \Box\beta} \qquad \frac{\alpha \Rightarrow \beta}{\rhd\beta \Rightarrow \rhd\alpha} \qquad \frac{\alpha \Rightarrow \beta}{\lhd\beta \Rightarrow \lhd\alpha}$$

Here x, y, z are arbitrary variables and α, β, γ are arbitrary terms.

Remark 2.2. This is clearly a generalization of classical modal logics: if we take for both $\rhd$ and $\lhd$ the classical negation $\neg$, then we may identify classical normal modal logics with those distributive modal logics containing the sequents: $\rhd\alpha \Rightarrow \lhd\alpha$, $\lhd\alpha \Rightarrow \rhd\alpha$, $\Diamond\alpha \Rightarrow \rhd\Box\rhd\alpha$, $\rhd\Box\rhd\alpha \Rightarrow \Diamond\alpha$, $\top \Rightarrow \alpha \vee \rhd\alpha$, and $\alpha \wedge \rhd\alpha \Rightarrow \bot$.

Definition 2.3. The minimal distributive modal logic will be called **DM**, and if Γ is a set of sequents we denote by **DM**.Γ the least distributive modal logic containing Γ.

Relational semantics

In the case of intuitionistic logic or Boolean modal logic, the relational or Kripke-style semantics has been an invaluable tool for analyzing logics. We believe that the relational semantics for distributive modal logics that we are about to develop will prove its use as well. It will come as no surprise that just as in the case of intuitionistic (modal) logic, Priestley duality for distributive lattices, or Celani and Jansana's semantics for positive modal logic, our relational structures carry, besides a binary relation for each of the modalities, also an ordering relation.

Definition 2.4. A *Kripke frame for distributive modal logic* (in short: *frame*) is a tuple $\mathbb{F} = (W, \leq, R_\Diamond, R_\Box, R_\rhd, R_\lhd)$ where W is a nonempty set, $\leq$ is a partial order on W, and $R_\Diamond, R_\Box, R_\rhd, R_\lhd$ are binary relations on W satisfying the following conditions (the symbol $\circ$ denotes relation composition):

$$\begin{aligned} &\leq \circ R_\Diamond \circ \leq \subseteq R_\Diamond \\ &\geq \circ R_\Box \circ \geq \subseteq R_\Box \\ &\geq \circ R_\rhd \circ \leq \subseteq R_\rhd \\ &\leq \circ R_\lhd \circ \geq \subseteq R_\lhd. \end{aligned} \tag{KF}$$

A *valuation* on a frame $\mathbb{F}$ is a map $V : X \to \mathcal{P}(W)$ from the set of variables to the power set of the domain of $\mathbb{F}$. Such a valuation is called *persistent* if $V(x)$ is downward closed for each variable x; that is, if $w \in V(x)$ and $v \leq w$ then $v \in V(x)$. A *Kripke model for distributive modal logic* (in short: *model*) is simply a pair $(\mathbb{F}, V)$ consisting of a frame $\mathbb{F}$ and a persistent valuation V on $\mathbb{F}$.

Given a model $\mathbb{M} = (\mathbb{F}, V)$ we define the *truth* relation $\Vdash$ between points and formulas by the following induction:

(1) For $x \in X$ we define $\mathbb{M}, w \Vdash x$ if and only if $w \in V(x)$;

(2) Suppose $\Vdash$ has been specified for terms α and β, then for each $w \in W$ we put
   (a) $\mathbb{M}, w \Vdash \alpha \vee \beta$ if and only if $\mathbb{M}, w \Vdash \alpha$ or $\mathbb{M}, w \Vdash \beta$;
   (b) $\mathbb{M}, w \Vdash \alpha \wedge \beta$ if and only if $\mathbb{M}, w \Vdash \alpha$ and $\mathbb{M}, w \Vdash \beta$;
   (c) $\mathbb{M}, w \Vdash \top$;
   (d) $\mathbb{M}, w \nVdash \bot$;
   (e) $\mathbb{M}, w \Vdash \Diamond\alpha$ if and only if there is $v \in W$ with $wR_{\Diamond}v$ and $\mathbb{M}, v \Vdash \alpha$;
   (f) $\mathbb{M}, w \Vdash \Box\alpha$ if and only if for all $v \in W$ with $wR_{\Box}v$ we have $\mathbb{M}, v \Vdash \alpha$;
   (g) $\mathbb{M}, w \Vdash \rhd\alpha$ if and only if for all $v \in W$ with $wR_{\rhd}v$ we have $\mathbb{M}, v \nVdash \alpha$;
   (h) $\mathbb{M}, w \Vdash \lhd\alpha$ if and only if there is $v \in W$ with $wR_{\lhd}v$ and $\mathbb{M}, v \nVdash \alpha$.

Definition 2.5. A model $\mathbb{M}$ satisfies a sequent $\alpha \Rightarrow \beta$, written $\mathbb{M} \Vdash \alpha \Rightarrow \beta$, provided for each $w \in W$ with $w \Vdash \alpha$ we have $w \Vdash \beta$. A frame $\mathbb{F}$ validates a sequent $\alpha \Rightarrow \beta$, written $\mathbb{F} \Vdash \alpha \Rightarrow \beta$, provided each model $(\mathbb{F}, V)$ satisfies $\alpha \Rightarrow \beta$. A frame $\mathbb{F}$ validates a set of sequents Γ, written $\mathbb{F} \Vdash \Gamma$, provided $\mathbb{F} \Vdash \alpha \Rightarrow \beta$ for each sequent $\alpha \Rightarrow \beta \in \Gamma$.

Definition 2.6. Given a class of frames C, let $\mathrm{Th}(\mathsf{C})$ be the set of sequents that are valid in (every frame of) C. Conversely, given a distributive modal logic Λ, let $\mathsf{Fr}(\Lambda)$ be the class of all frames validating Λ.

It is easy to see that the maps Th and Fr form a Galois connection between classes of frames and sets of formulas. Just as in the case of classical modal logic, the stable sets/classes will be of interest. First we define the notions of soundness and completeness.

Definition 2.7. A distributive modal logic Λ is called *sound* with respect to a class C of frames if $\Lambda \subseteq \mathrm{Th}(\mathsf{C})$ and *complete with respect to* C if, conversely, $\mathrm{Th}(\mathsf{C}) \subseteq \Lambda$. Λ is called *complete* if it is complete with respect to the class $\mathsf{Fr}(\Lambda)$.

Equivalently, Λ is complete iff it is stable with respect to the Galois connection mentioned above; that is, if $\Lambda = \mathrm{Th}(\mathsf{Fr}(\Lambda))$. In the other direction, we will call a class C of frames *definable* if is stable, that is, if $\mathsf{C} = \mathsf{Fr}(\mathrm{Th}(\mathsf{C}))$, or, equivalently, if C is the frame class of some set of formulas. We are particularly interested in those definable classes that are elementary (that is, definable in first order logic), and introduce the notion of correspondence for modal sequents.

Definition 2.8. Let $\alpha \Rightarrow \beta$ be a modal sequent and let ϕ be a formula in the first order language of frames for distributive modal logic. We say $\alpha \Rightarrow \beta$ and ϕ *correspond* to each other if for all frames $\mathbb{F}$, we have that $\mathbb{F} \Vdash \alpha \Rightarrow \beta$ iff $\mathbb{F} \Vdash \phi$. Similar definitions apply to sets of sequents and sets of first order formulas.

It is obvious that the above notions are just two out of a vast array of interesting concepts. Very shortly we will add a third one, *canonicity*, to the repertoire, but we prefer to introduce that topic algebraically.

2.2. Distributive modal algebras

Introduction

We will now set out to describe the *algebraic* semantics for distributive modal logic. Considering each sequent $\alpha \Rightarrow \beta$ as an algebraic inequality $\alpha \preccurlyeq \beta$, or equivalently as an algebraic identity $\alpha \wedge \beta \approx \alpha$, we see readily that the distributive modal logics are algebraic.

Thus they are the equational theories of corresponding varieties of algebras, cf. the next subsection for more details.

The crucial concept on the algebraic side is that of a distributive modal algebra.

Definition 2.9. A *distributive modal algebra* (DMA) is an algebra $\mathbb{A} = (A, \vee, \wedge, \bot, \top, \Diamond, \Box, \rhd, \lhd)$ where $(A, \vee, \wedge, \bot, \top)$ is a DL (that is, a bounded distributive lattice) and the additional operations (called *modal operators*) satisfy

$$\begin{array}{ll} \Diamond(x \vee y) \approx \Diamond x \vee \Diamond y & \Diamond\bot \approx \bot \\ \Box(x \wedge y) \approx \Box x \wedge \Box y & \Box\top \approx \top \\ \rhd(x \vee y) \approx \rhd x \wedge \rhd y & \rhd\bot \approx \top \\ \lhd(x \wedge y) \approx \lhd x \vee \lhd y & \lhd\top \approx \bot. \end{array}$$

The structure $\mathbb{D}_{\mathbb{A}} = (A, \vee, \wedge, \bot, \top)$ is called the *underlying lattice* of $\mathbb{A}$, and it is sometimes convenient to write $\mathbb{A} = (\mathbb{D}_{\mathbb{A}}, \Diamond, \Box, \rhd, \lhd)$, or $\mathbb{A} = (\mathbb{D}, \Diamond, \Box, \rhd, \lhd)$.

It is important to realize that one significant difference between this setting and Boolean modal logic is that here, operators of one type are not definable in terms of operators of another type. (In the Boolean case it is easy to see that for instance $\rhd$ can be seen as an operation of the form $\neg\Diamond_{\rhd}$, with $\Diamond_{\rhd}$ being defined by $\Diamond_{\rhd} a = \neg\rhd a$.) On the other hand it is clear that the four types of operators defined above will display very similar behavior. In order to make this precise, we will exploit the well-known duality principle which is based on the fact that the class of distributive lattices is closed under taking *order duals*. To be well equipped for this work we need some terminology.

Definition 2.10. Given a DL $\mathbb{A} = (A, \vee, \wedge, \bot, \top)$, we let $\mathbb{A}^{\partial}$ denote the *dual* lattice, that is, the structure $\mathbb{A}^{\partial} = (A, \wedge, \vee, \top, \bot)$. For technical convenience, we define $\mathbb{A}^1 = \mathbb{A}$.

An element $\varepsilon \in \{1, \partial\}^n$ is called an *order type*; the i-th component of such an ε will be denoted ε_i. Given an order type $\varepsilon \in \{1, \partial\}^n$, we let $\mathbb{A}^{\varepsilon}$ denote the algebra $\mathbb{A}^{\varepsilon_1} \times \cdots \times \mathbb{A}^{\varepsilon_n}$.

Remark 2.11. Observe that for any $\varepsilon \in \{1, \partial\}^n$, the two algebras $\mathbb{A}^n$ and $\mathbb{A}^{\varepsilon}$ are based on the same domain, A^n. When we write 'let $f : \mathbb{A}^n \to \mathbb{B}$ be a map' we implicitly understand that the domain $\mathbb{A}^n$ and codomain $\mathbb{B}$ form part of the definition of f.

For instance, suppose that, given two order types ε and ε', we consider two maps $f : \mathbb{A}^{\varepsilon} \to \mathbb{B}$ and $g : \mathbb{A}^{\varepsilon'} \to \mathbb{B}$ that are set-theoretically identical (that is, $f(\overline{a}) = g(\overline{a})$ for all $\overline{a} \in A^n$). Such maps will be called *order variants*. For the time being, we will not identify order variants that are based on differently ordered domains. The advantage of this is that, for instance, it allows us to say that f is order preserving while g is not.

Canonical extensions of distributive lattices

Since varieties of DMAs correspond to distributive modal logics, in order to prove completeness results algebraically we will need to make some kind of link between DMAs and frames. Modal logicians are used to obtaining a frame from a distributive modal algebra using some kind of duality theory, usually based on either Stone duality for Boolean algebras or Priestley duality for distributive lattices (see e.g. [18]). Here we will work differently; instead of working with the dual frame directly, we will describe it algebraically as the *perfect* or *canonical extension* of the original algebra, in the tradition of Jónsson and Tarski's original papers [21,22]. In the next subsection then, we will see how to use a different kind of duality to obtain a distributive modal frame from a DMA. The advantage of this approach, which to a modal logician may seem rather roundabout at first, is that it allows us to apply the powerful algebraic theory concerning canonical extensions. In the interest of self-containment we give here the basic definitions needed and prove some key technical results. For other facts, see [1,5,9], and for a more in-depth treatment of canonical extensions, see [12–15].

We start with the definition of the canonical extension of a bounded distributive lattice (DL). This algebra will be a special distributive lattice in which the original lattice is embedded in a very specific manner.

Definition 2.12. Suppose $\mathbb{A}$ is a (bounded) sublattice of a complete lattice $\mathbb{A}'$. We say that

(1) $\mathbb{A}$ is *dense* in $\mathbb{A}'$ if every element of A' can be expressed both as a join of meets and as a meet of joins of elements from A.
(2) $\mathbb{A}$ is *compact* in $\mathbb{A}'$ if, for all $S, T \subseteq A$ with $\bigwedge S \leq \bigvee T$ in $\mathbb{A}'$, there exist finite sets $F \subseteq S$ and $G \subseteq T$ such that $\bigwedge F \leq \bigvee G$.

It can be shown that, given a DL $\mathbb{A}$, one can always find a complete lattice that satisfies these density and compactness conditions. For instance, readers familiar with Priestley duality [10] could check that the *double dual*, that is, the lattice of down-sets of the order of the Priestley dual of the lattice, is an instance. Besides this existence condition there is also a uniqueness claim: one can show that between any two complete DL extensions of $\mathbb{A}$ that satisfy these two conditions, there must be a unique isomorphism, which reduces to the identity when restricted to $\mathbb{A}$. This justifies and motivates the following definition.

Definition 2.13. The canonical extension of a DL $\mathbb{A}$ is a complete lattice $\mathbb{A}^\sigma$ containing $\mathbb{A}$ as a dense and compact sublattice.

These seemingly very weak conditions have a very strong impact on the properties of $\mathbb{A}^\sigma$. To start with, it is a *perfect* distributive lattice:

Definition 2.14. A distributive lattice $\mathbb{A}$ is called *perfect* or a DL$^+$ if it satisfies one of the following, equivalent, conditions:

(1) $\mathbb{A}$ is doubly algebraic (that is, both $\mathbb{A}$ and $\mathbb{A}^\partial$ are algebraic),
(2) $\mathbb{A}$ is complete, completely distributive and join generated by the set $J^\infty(\mathbb{A})$ of all completely join irreducible elements of $\mathbb{A}$ (as well as meet generated by the set $M^\infty(\mathbb{A})$ of all completely join irreducible elements of $\mathbb{A}$),
(3) $\mathbb{A}$ is isomorphic to a set-theoretic lattice based on the collection of down-sets of some partial order.

Concerning the connection between $\mathbb{A}$ and $\mathbb{A}^\sigma$, the density implies that $J^\infty(\mathbb{A}^\sigma)$ is contained in the meet closure $K(\mathbb{A}^\sigma)$ of $\mathbb{A}$ in $\mathbb{A}^\sigma$ and that $M^\infty(\mathbb{A}^\sigma)$ is contained in the join closure $O(\mathbb{A}^\sigma)$ of $\mathbb{A}$ in $\mathbb{A}^\sigma$. As mentioned above, one way to obtain the canonical extension of a DL is to take the poset of all order filters, or up-sets, of its topological dual space. For this reason we refer to elements of $\mathbb{A}$ as clopen elements of $\mathbb{A}^\sigma$, to elements of $K(\mathbb{A}^\sigma)$ as closed elements, and to elements of $O(\mathbb{A}^\sigma)$ as open elements.

Taking the canonical extension of a DL is an operation that interacts nicely with taking order duals or products. That is, we have

$$(\mathbb{A}^{\partial})^{\sigma} \cong (\mathbb{A}^{\sigma})^{\partial},$$
$$(\mathbb{A}^{n})^{\sigma} \cong (\mathbb{A}^{\sigma})^{n},$$

and, as a consequence,

$$(\mathbb{A}^{\varepsilon})^{\sigma} \cong (\mathbb{A}^{\sigma})^{\varepsilon},$$

for every order type ε. Also, the operation $(\cdot)^{\partial}$ interchanges closed and open elements: $K(\mathbb{A}^{\partial\sigma}) = O(\mathbb{A}^{\sigma})^{\partial}$ and $O(\mathbb{A}^{\partial\sigma}) = K(\mathbb{A}^{\sigma})^{\partial}$; similarly, we have that $K(\mathbb{A}^{n\sigma}) = (K(\mathbb{A}^{\sigma}))^{n}$, and $O(\mathbb{A}^{n\sigma}) = (O(\mathbb{A}^{\sigma}))^{n}$. In the sequel, we will identify $(\mathbb{A}^{\partial})^{\sigma}$ with $(\mathbb{A}^{\sigma})^{\partial}$, $(\mathbb{A}^{n})^{\sigma}$ with $(\mathbb{A}^{\sigma})^{n}$, and $(\mathbb{A}^{\varepsilon})^{\sigma}$ with $(\mathbb{A}^{\sigma})^{\varepsilon}$, for any order type ε.

Extending maps

The canonical extension of a distributive modal algebra $\mathbb{A} = (A, \vee, \wedge, \bot, \top, \Diamond, \Box, \rhd, \lhd)$ will obviously be an expansion of the canonical extension of the underlying lattice $(A, \vee, \wedge, \bot, \top)$. In order to extend operations on a DL $\mathbb{A}$ to its canonical extension, we will consider the case of maps $f : \mathbb{A} \to \mathbb{B}$ between DLs. (Note that this suffices by the earlier made identification of $(\mathbb{A}^{n})^{\sigma}$ with $(\mathbb{A}^{\sigma})^{n}$.)

The idea in extending maps is that the value of the extension at a point of the canonical extension of the domain depends on the value of the original map at ‘nearby’ points. This can be made precise by introducing a topology on the domain, as was done in [15]. In that setting one can then see the two extensions given below as the lower and upper envelope of the original map. Here we give the definition without explicit reference to topology; following the definition we provide some further explanation.

Definition 2.15. Given a map $f : \mathbb{A} \to \mathbb{B}$ between DLs, we define two maps $f^{\sigma}, f^{\pi} : \mathbb{A}^{\sigma} \to \mathbb{B}^{\sigma}$ by

$$f^{\sigma}(u) = \bigvee \left\{ \bigwedge \{ f(a) : a \in A \text{ and } x \le a \le y \} : K(\mathbb{A}^{\sigma}) \ni x \le u \le y \in O(\mathbb{A}^{\sigma}) \right\}$$
$$f^{\pi}(u) = \bigwedge \left\{ \bigvee \{ f(a) : a \in A \text{ and } x \le a \le y \} : K(\mathbb{A}^{\sigma}) \ni x \le u \le y \in O(\mathbb{A}^{\sigma}) \right\}.$$

The idea of the definition is based on an approximation of a point u in $\mathbb{A}^{\sigma}$ by *intervals* of the form $[x, y]$ with x a closed element below u and y an open one above u. The ‘nearby’ points that we mentioned earlier on are the clopen elements of such intervals; note that since f is defined on A we already have f-values for these clopen elements. For the definition of f^{π}, the contribution of the interval $[x, y]$ is valued as the *join* $\bigvee_{a\in[x,y]_A} f(a)$, where $[x, y]_A = \{a \in A \mid x \le a \le y\}$ denotes the set of clopens in $[x, y]$. It is easy to see that this join will shrink when we move x and y closer to u—simply because we have less joinands. Therefore, it makes sense to define $f^{\pi}(u)$ as the *meet* of the values obtained from all these closed-open intervals surrounding u. In particular, since clopen elements are both closed and open, it is not difficult to show that this definition makes f^{π} really an *extension* of f. Obviously, the definition of f^{σ} can be explained dually.

Remark 2.16. It is important to realize how much of the definition of these extensions of a map f depends on the structure of the domain and of the codomain. What we mean is the following.

Let $\mathbb{A}$ and $\mathbb{B}$ be DLs, and let $f : \mathbb{A}^n \to \mathbb{B}$ be a map. Observe that the definition of f^σ depends on the lattice structure of $\mathbb{A}^n$. To make this more precise, consider an order variant $g : \mathbb{A}^\varepsilon \to \mathbb{B}$ of f (cf. Remark 2.11). Now the definition of g^σ will depend on the lattice structure of $\mathbb{A}^\varepsilon$ which is *different* from that of $\mathbb{A}^n$! So perhaps f and g will be distinct, even as set-theoretical maps?

No. A closer inspection of the definition of the σ-extension reveals a self-duality in terms of the domain order. Using this self-duality, one can easily prove that if f and g are order variants, then so are f^σ and g^σ; in particular, f^σ and g^σ are identical as set-theoretical maps. Of course the same holds for π-extensions. For this reason in the sequel we will not give different names to order variant maps.

On the other hand, dualizing the order on the codomain *does* make a difference. To this end, given a map $f : A \to B$, we define $f^\partial : A^\partial \to B^\partial$ to be the same set theoretic map as f but with the order dualized on both domain and codomain. Then it is easy to see from the definitions of σ- and π-extensions of maps that $(f^\partial)^\sigma = (f^\pi)^\partial$ and that $(f^\partial)^\pi = (f^\sigma)^\partial$.

In case the original map is order preserving we can simplify these descriptions:

Remark 2.17. If the DL map $f : \mathbb{A} \to \mathbb{B}$ is order preserving, then for all $u \in A^\sigma$,

$$f^\sigma(u) = \bigvee \left\{ \bigwedge\{f(a) : x \le a \in A\} : u \ge x \in K(\mathbb{A}^\sigma) \right\},$$
$$f^\pi(u) = \bigwedge \left\{ \bigvee\{f(a) : x \le a \in A\} : u \le y \in O(\mathbb{A}^\sigma) \right\}.$$

Also, for all $x \in K(\mathbb{A}^\sigma)$ and $y \in O(\mathbb{A}^\sigma)$,

$$\begin{aligned} f^\sigma(x) &= f^\pi(x) = \bigwedge\{f(a) : x \le a \in A\}, \\ f^\sigma(y) &= f^\pi(y) = \bigvee\{f(a) : x \le a \in A\}. \end{aligned}$$

In particular, f^σ and f^π *agree* on closed and open elements, and both operations map closed elements to closed elements, and opens to opens.

But also in the general setting, these lower and upper extensions have special properties. These can best be expressed in topological terms. Without going into the details concerning the topologies involved, we just note that f^σ is the largest 'upper continuous' (UC) extension of f, and f^π is the least 'lower continuous' (LC) extension of f:

Theorem 2.18. *Given a map $f : \mathbb{A} \to \mathbb{B}$ between DLs, the map $f^\sigma : \mathbb{A}^\sigma \to \mathbb{B}^\sigma$ is an extension of f. In fact, f^σ is the largest extension of f to $\mathbb{A}^\sigma$ satisfying:*

(UC) *For all $u \in A^\sigma$ and for all $q \in J^\infty(\mathbb{B}^\sigma)$, if $q \le f^\sigma(u)$ then there exist $K(\mathbb{A}^\sigma) \ni x \le u \le y \in O(\mathbb{A}^\sigma)$ so that $q \le f^\sigma(v)$ for all $x \le v \le y$.*

Here if f is order preserving then so is f^σ and the $y \in O(\mathbb{A}^\sigma)$ is not necessary.

Similarly, the map $f^\pi : \mathbb{A}^\sigma \to \mathbb{B}^\sigma$ is also an extension of f. In fact, f^π is the smallest extension of f to $\mathbb{A}^\sigma$ satisfying:

(LC) *For all* $u \in A^\sigma$ *and for all* $n \in M^\infty(\mathbb{B}^\sigma)$, *if* $n \geq f^\pi(u)$ *then there exist* $K(\mathbb{A}^\sigma) \ni x \leq u \leq y \in O(\mathbb{A}^\sigma)$ *so that* $n \geq f^\pi(v)$ *for all* $x \leq v \leq y$.

Here if f *is order preserving then so is* f^π *and the* $x \in K(\mathbb{A}^\sigma)$ *is not necessary.*

Canonical extensions of distributive modal algebras

Depending on the nature of a map or operation it is convenient to use either its σ- or π-extension. In general, for maps that send joins or meets in the domain to joins in the codomain, it is advantageous to pick the σ-extension as this one will then send arbitrary joins or meets in the domain to arbitrary joins in the codomain. Dually for maps sending joins or meets in the domain to meets in the codomain, it is usually more advantageous to work with the π-extension. Maps for which the two extensions agree display, as one would expect, particularly nice behaviour. These are called *smooth*. All maps that preserve either joins or meets, and all maps that turn joins into meets, or vice versa, are smooth. Notice that all the basic operations of a DMA are of this kind. As a consequence, the following definition is unambiguous.

Definition 2.19. Let $\mathbb{A} = (\mathbb{D}, \Diamond, \Box, \rhd, \lhd)$ be a distributive modal algebra. The *canonical* or *perfect extension of* $\mathbb{A}$ is the algebra $\mathbb{A}^\sigma = (\mathbb{D}^\sigma, \Diamond^\sigma, \Box^\sigma, \rhd^\sigma, \lhd^\sigma) = (\mathbb{D}^\sigma, \Diamond^\pi, \Box^\pi, \rhd^\pi, \lhd^\pi)$.

Just as in the case of distributive lattices without additional structure, these canonical extensions satisfy some additional nice properties.

Definition 2.20. A distributive modal algebra $\mathbb{A} = (\mathbb{D}, \Diamond, \Box, \rhd, \lhd)$ is called *perfect* or a DMA$^+$ if $\mathbb{D}$ is a perfect distributive lattice, while the modal operators satisfy the following infinitary distribution properties:

$$\Diamond\left(\bigvee X\right) \approx \bigvee \Diamond(X)$$
$$\Box\left(\bigwedge X\right) \approx \bigwedge \Box(X)$$
$$\rhd\left(\bigvee X\right) \approx \bigwedge \rhd(X)$$
$$\lhd\left(\bigwedge X\right) \approx \bigvee \lhd(X).$$

Observe that the distributive laws of Definition 2.20 are generalizations of the finitary DMA laws for distributive modalities to the case of infinite joins and meets. Perfect DMAs are fairly nice structures to work with: below we will see that they are in fact distributive modal frames in algebraic disguise. That also explains the logical importance of canonical extensions.

Lemma 2.21. *If* $\mathbb{A}$ *is a* DMA, *then* $\mathbb{A}^\sigma$ *is a* DMA$^+$.

Proof. The fact that the underlying lattice of $\mathbb{A}^\sigma$ is perfect was first proved in [13, Theorem 2.3], and the fact that $\Diamond(\bigvee X) \approx \bigvee \Diamond(X)$ holds in $\mathbb{A}^\sigma$ is the content of [13, Lemma 2.5(iii)]. The properties of the other operations follow using the appropriate order duals. $\Box$

All of this has as consequence that properties that are preserved under moving to the canonical extension of a distributive modal algebra are of great interest. Such properties will be called *canonical*.

Definition 2.22. A class of distributive modal algebras is *canonical* if it is closed under taking canonical extensions. An equation, formula, or set of formulas, is called *canonical*, if the class of DMAs defined by the equation, formula, or set of formulas, is canonical.

2.3. Logic and algebra

In this section we establish some links between the area of distributive modal logic and that of distributive modal algebra. We will assume familiarity with the basics of algebraic logic (such as outlined in [4]), and concentrate on some specific issues. As usual, there will be two *kinds* of links between logic and algebra: on a syntactic level, we will see that the notion of a distributive modal logic corresponds to that of a *variety* of distributive modal algebras. The other link is between the logical and algebraic *structures*. We will start with this second connection here. That is, we will see how to obtain algebras from frames and vice versa.

Frames and perfect distributive modal algebras

Just as in the case of classical modal logic, there are various ways to move from frames to algebras and back. The operations that we present here arise naturally if we combine dualities originating with Birkhoff [3] and Thomason [27]. Birkhoff duality is a duality between finite distributive lattices and finite posets: given a finite poset, the down-sets form a finite distributive lattice, and given a finite distributive lattice the join irreducible elements form a finite poset. Of course, a generalization we will need here is to remove the finiteness restriction; when doing this one must restrict oneself to DL^+s, but one can keep arbitrary posets. In order to deal with the additional operations on the lattice, we generalize Thomason's duality between frames for classical modal logic and complete and atomic Boolean algebras with completely additive operators. From the perspective of Birkhoff duality, this means that the dual objects will thus be endowed with additional relations.

Now we turn to the technical details. For a frame $\mathbb{F} = (W, \leq, R_\Diamond, R_\Box, R_\rhd, R_\lhd)$, let $\mathcal{D}(W)$ be the collection of all downward closed sets (or, simply, *down-sets*) of W. Recall that a subset S of W is downward closed if $u \leq v \in S$ implies $u \in S$. Now consider, for a relation R on W, the following operations on subsets of W:

$$
\begin{aligned}
\langle R\rangle\, S &:= \{u : \exists v (uRv \text{ and } v \in S)\}\\
[R]\, S &:= \{u : \forall v (uRv \to v \in S)\} \quad (= -\langle R\rangle - S)\\
[R\rangle\, S &:= \{u : \forall v (uRv \to v \notin S)\} \quad (= -\langle R\rangle\, S)\\
\langle R]\, S &:= \{u : \exists v (uRv \text{ and } v \notin S)\} \quad (= \langle R\rangle - S).
\end{aligned}
$$

We leave it for the reader to verify that $\mathcal{D}(W)$ is closed under the operations $\langle R_\Diamond\rangle, [R_\Box], [R_\rhd\rangle$ and $\langle R_\lhd]$; this follows from the conditions (KF) that any distributive modal frame satisfies by Definition 2.4. Hence, we may correctly define the *dual* or *complex algebra* of $\mathbb{F}$ as the structure

$$\mathbb{F}^+ = (\mathcal{D}(W), \cup, \cap, \emptyset, W, \langle R_\Diamond\rangle, [R_\Box], [R_\rhd\rangle, \langle R_\lhd]).$$

We leave the straightforward verification that $\mathbb{F}^+$ is in fact a *perfect* distributive modal algebra as an exercise for the reader.

In order to be able to relate our algebraic results to questions of frame completeness we need the following lemma:

Lemma 2.23. *Let $\mathbb{F}$ be a frame and $\alpha \Rightarrow \beta$ a sequent. Then $\mathbb{F} \Vdash \alpha \Rightarrow \beta$ if and only if $\mathbb{F}^+ \models \alpha \preccurlyeq \beta$.*

Proof. Straightforward from the definitions: for instance, observe that a persistent valuation on the frame $\mathbb{F}$ is nothing but an assignment on the algebra $\mathbb{F}^+$. $\square$

Conversely, for a DMA$^+$ $\mathbb{A} = (\mathbb{D}, \Diamond, \Box, \rhd, \lhd)$, let $J^\infty(\mathbb{A})$ (resp. $M^\infty(\mathbb{A})$) be the set of completely join-irreducible (resp. meet-irreducible) elements of the (perfect!) lattice reduct $\mathbb{D}$ of $\mathbb{A}$. Define binary relations on $J^\infty(\mathbb{A})$ by

$$\begin{array}{lll} u R_\Diamond v & \text{iff} & u \leq \Diamond v \\ u R_\Box v & \text{iff} & \kappa(u) \geq \Box \kappa(v) \\ u R_\rhd v & \text{iff} & \kappa(u) \geq \rhd v \\ u R_\lhd v & \text{iff} & u \leq \lhd \kappa(v) \end{array} \tag{DR}$$

where $\kappa : J^\infty(\mathbb{A}) \to M^\infty(\mathbb{A})$ is the order isomorphism defined by $\kappa(u) = \bigvee(- \uparrow u)$. The dual of $\mathbb{A}$, which we will call the *atom structure* of $\mathbb{A}$, is defined to be the structure

$$\mathbb{A}_+ = (J^\infty(\mathbb{A}), \leq, R_\Diamond, R_\Box, R_\rhd, R_\lhd),$$

where $\leq$ is the order on the lattice A restricted to $J^\infty(\mathbb{A})$. Then one can show that $\mathbb{A}_+$ satisfies (KF) given in Definition 2.4. Thus $\mathbb{A}_+ = (J^\infty(\mathbb{A}), \leq, R_\Diamond, R_\Box, R_\rhd, R_\lhd)$ is a Kripke frame for distributive modal logic.

Going back and forth between frames and algebras, we can prove the following results.

Proposition 2.24. *Let $\mathbb{F}$ be a frame for distributive modal logic. Then $(\mathbb{F}^+)_+ \cong \mathbb{F}$.*

To prove this, notice that the completely join irreducible elements of the lattice $\mathcal{D}(W)$ are exactly the principal up-sets $\uparrow u$ for $u \in W$. The fact that this correspondence is an isomorphism of relational structures is fairly straightforward to check.

Proposition 2.25. *Let $\mathbb{A}$ be a* DMA$^+$. *Then, $\mathbb{A} \cong (\mathbb{A}_+)^+$.*

To prove this, notice that the DL isomorphism $\eta : \mathbb{D}_\mathbb{A} \to (\mathbb{D}_{\mathbb{A}+})^+$ is given by $\eta(a) = J^\infty(\mathbb{A}) \cap \downarrow a$. The crucial part is that η preserves the modal operators, which we state separately as follows:

Lemma 2.26. *Let $\mathbb{A} = (\mathbb{D}_\mathbb{A}, \Diamond, \Box, \rhd, \lhd)$ be a* DMA$^+$ *and $\mathbb{A}_+ = (J^\infty(\mathbb{A}), \leq, R_\Diamond, R_\Box, R_\rhd, R_\lhd)$ its dual frame. For all $a \in A$ and all $u \in J^\infty(\mathbb{A})$,*

(1) $u \leq \Diamond a$ *iff* $\exists v(u R_\Diamond v$ *and* $v \leq a)$
(2) $u \leq \Box a$ *iff* $\forall v(u R_\Box v \to v \leq a)$
(3) $u \leq \rhd a$ *iff* $\forall v(u R_\rhd v \to v \not\leq a)$
(4) $u \leq \lhd a$ *iff* $\exists v(u R_\lhd v$ *and* $v \not\leq a)$.

Proof. To show the first statement, suppose $u \leq \Diamond a$. As

$$\begin{aligned}\Diamond a &= \Diamond\left(\bigvee (J^{\infty}(\mathbb{A}) \cap \downarrow a)\right)\\ &= \bigvee (\Diamond (J^{\infty}(\mathbb{A}) \cap \downarrow a))\end{aligned}$$

and u is completely join-prime, there is $v \in J^{\infty}(\mathbb{A}) \cap \downarrow a$ with $u \leq \Diamond v$. This means $\exists v (u R_{\Diamond} v$ and $v \leq a)$. The converse is trivial.

To prove the second statement suppose $u \not\leq \Box a$. Recall that $\kappa(u) = \bigvee (A - \downarrow u)$, that is, $u \not\leq x$ if and only if $\kappa(u) \geq x$ for $x \in A$. Thus

$$\begin{aligned}\Box a &= \Box \bigwedge (M^{\infty}(\mathbb{A}) \cap \uparrow a)\\ &= \bigwedge (\Box (M^{\infty}(\mathbb{A}) \cap \uparrow a)).\end{aligned}$$

Now since $\kappa(u)$ is completely meet-prime, it follows that there is $m \in M^{\infty}(\mathbb{A}) \cap \uparrow a$ with $\kappa(u) \geq \Box m$. Let $v \in J^{\infty}(\mathbb{A})$ with $\kappa(v) = m$. Then we have $a \leq \kappa(v)$ and $\kappa(u) \geq \Box \kappa(v)$, that is, $v \not\leq a$ and $u R_{\Box} v$. So $u \not\leq \Box a$ implies $\exists v (u R_{\Box} v$ and $v \not\leq a)$. Conversely, if there is $v \in J^{\infty}(\mathbb{A})$ with $u R_{\Box} v$ and $v \not\leq a$, then $a \leq \kappa(v)$, so $\Box a \leq \Box \kappa(v) \leq \kappa(u)$, and therefore $u \not\leq \Box a$.

The other statements are proved similarly. $\Box$

This connection between frames and perfect DMAs can in fact be made into a full categorical duality but this is not our concern here.

Logics and varieties

Given a set Γ of modal sequents, let $\Gamma^{\preccurlyeq}$ denote the set $\{\alpha \preccurlyeq \beta \mid \alpha \Rightarrow \beta \in \Gamma\}$ of *corresponding inequalities*. Since any inequality is (or can be seen as) an equation, the following definition makes sense.

Definition 2.27. For a distributive modal logic Λ, let V_{Λ} be the variety defined by the set $\Lambda^{\preccurlyeq}$. V_{Λ} is called the *variety corresponding to* Λ.

Without proof we mention the following result, which can be obtained by standard techniques from algebraic logic.

Proposition 2.28. *The operation $\Lambda \mapsto \mathsf{V}_{\Lambda}$ mapping distributive modal logics to their corresponding varieties is an isomorphism between the lattice of distributive modal logics and the lattice of varieties of modal algebras.*

This motivates the following definition.

Definition 2.29. A distributive modal logic Λ is called *canonical* if the corresponding set $\Lambda^{\preccurlyeq}$ of inequalities is canonical in the algebraic sense.

Our logical interest in the notion of canonicity stems from the following result.

Lemma 2.30. *Let Λ be a distributive modal logic. If Λ is canonical, then Λ is complete.*

Proof. For a distributive modal logic Λ, let $\mathbb{A}_{\Lambda}$ be the free algebra over a countably infinite set of propositional variables, in the variety corresponding to Λ; logicians will know $\mathbb{A}_{\Lambda}$

under the name *Lindenbaum–Tarski* algebra. We leave it to the reader to verify that for all modal sequents $\alpha \Rightarrow \beta$ we have the following equivalence:

$$\alpha \Rightarrow \beta \in \Lambda \iff \mathbb{A}_\Lambda \models \alpha \preccurlyeq \beta. \qquad (*)$$

This equivalence is exactly what it means for $\mathbb{A}_\Lambda$ to be free.

Now let $\alpha \Rightarrow \beta$ be a modal sequent. Then we have

$$\begin{aligned} \alpha \Rightarrow \beta \notin \Lambda &\iff \mathbb{A}_\Lambda \not\models \alpha \preccurlyeq \beta \\ &\implies \mathbb{A}_\Lambda^\sigma \not\models \alpha \preccurlyeq \beta \\ &\iff ((\mathbb{A}_\Lambda^\sigma)_+)^+ \not\models \alpha \preccurlyeq \beta \\ &\iff (\mathbb{A}_\Lambda^\sigma)_+ \not\Vdash \alpha \Rightarrow \beta. \end{aligned}$$

Here the first equivalence is a rephrasing of $(*)$; the implication follows from the fact that $\mathbb{A}_\Lambda$ is a subalgebra of $\mathbb{A}_\Lambda^\sigma$; the penultimate equivalence follows from Lemma 2.21 and Proposition 2.25; and the last equivalence is exactly the content of Lemma 2.23.

All in all we find that every non-theorem of Λ is refuted on the frame $(\mathbb{A}_\Lambda^\sigma)_+$. This justifies the definition of $\mathbb{C}_\Lambda$, the *canonical frame*, as the structure $(\mathbb{A}_\Lambda^\sigma)_+$. What we have just proved, then, is that every distributive modal logic is complete with respect to the singleton class consisting of its canonical frame.

Now assume that Λ is canonical. Then it follows from $(*)$ and the canonicity of $\Lambda^\preccurlyeq$ that $\mathbb{A}_\Lambda^\sigma \models \Lambda^\preccurlyeq$, so by the Lemmas 2.21 and 2.23 and Proposition 2.25 again, we find that $\mathbb{C}_\Lambda \Vdash \Lambda$. This means that any non-theorem $\alpha \Rightarrow \beta$ of Λ can actually be refuted *on a* Λ*-frame*, and thus shows that $\mathrm{Th}(\mathsf{Fr}(\Lambda)) \subseteq \Lambda$. In other words, it proves the completeness of Λ. $\square$

3. Sahlqvist sequents and inequalities

It is clear that not all distributive modal logics are canonical since it is not even so for classical modal logics. We now develop a notion of Sahlqvist sequents which is the most general possible while still restricting to the original definition in the classical modal logic setting. In the classical setting, when describing admissible Sahlqvist formulas it is convenient to work with a normal form in which all occurrences of negation ($\neg$) are moved right in front of the variables. In our generalized setting this is no longer available. Again, in the classical setting, the main feature that may make non-Sahlqvist formulas ill-behaved is that the 'outside' connectives are 'universal' (boxes), while the 'inside' connectives are 'choice connectives' (that is, diamonds or disjunction). These are also the features we will need to capture in the generalized setting. Due to the necessity of considering nestings of connectives including $\rhd$ and $\lhd$, however, our description is somewhat more complex.

We assume that the reader is familiar with the notion of a generation tree of a formula or term. (Our definitions below will refer to DML-formulas but obviously apply to DMA-terms as well.) Here, with each formula we will associate *two* generation trees; each of these will be an expansion of the formula's generation tree in which every node is signed with either $+$ (plus) or $-$ (minus). These signings are required to satisfy the following constraints:

- If a node is $\vee$, $\wedge$, $\Diamond$, or $\Box$, assign the same sign to its successor nodes.
- If a node is $\rhd$, or $\lhd$, assign the opposite sign to its successor node.

Note that by these conditions, the sign of each node is determined by the *initial condition*, that is, the sign of the root of the tree. It is thus that with each formula we may associate two *signed* generation trees, the *positive* and the *negative* one.

Definition 3.1. A node in a signed generation tree of a DML formula is said to be

(1) positive if it is signed '$+$' and negative if it is signed '$-$';
(2) a *choice* node if it is either positive and labelled $\vee, \Diamond, \lhd$ or negative and labelled $\wedge, \Box, \rhd$;
(3) *universal* if it is either positive and labelled $\Box$ or $\rhd$, or negative and labelled $\Diamond$ or $\lhd$.

Definition 3.2. A DML formula is said to be

(1) *uniform* provided, in a signed generation tree for the formula, there is no one variable that occurs both with a minus and a plus;
(2) *left* (resp. *right*) *universal* if it is uniform and there are no choice nodes in the positive (resp. negative) generation tree;
(3) *left* (resp. *right*) *Sahlqvist* if it is uniform and there are no choice nodes in the scope of universal nodes in the positive (resp. negative) generation tree.

Remark 3.3. It would be instructive but not exactly straightforward to make a precise comparison with the classical case. One of the obstacles is that in the classical case, there is a multitude of definitions of what a Sahlqvist formula is; however, we can take this situation to our advantage by selecting a definition that is most convenient for our present purposes. So, let us work with (a slight adaptation of) the definition given in [2]:

> Let ϕ and ψ be classical modal formulas in normal form; that is, built up from variables and negations of variables, using $\top, \bot, \wedge, \vee, \Box$ and $\Diamond$. Then the formula $\phi \to \psi$ is a *Sahlqvist–van Benthem* formula if (i) no positive occurrence of a variable in ϕ is in the scope of a $\Diamond$ or a $\vee$ which itself is in the scope of a $\Box$, and (ii) no negative occurrence of a variable in ψ is in the scope of a $\Box$ or an $\wedge$ which itself is in the scope of a $\Diamond$.

Now if we want to compare our Definition 3.2 to van Benthem's, we should first look at the following straightforward translation of a distributive modal logic formula to a classical one:

$$
\begin{array}{lcllcl}
tr(x_i) & = & x_i & & & \\
tr(\bot) & = & \bot & tr(\top) & = & \top \\
tr(\alpha \vee \beta) & = & tr(\alpha) \vee tr(\beta) & tr(\alpha \wedge \beta) & = & tr(\alpha) \wedge tr(\beta) \\
tr(\Diamond\alpha) & = & \Diamond tr(\alpha) & tr(\Box\alpha) & = & \Box tr(\alpha) \\
tr(\rhd\alpha) & = & \neg\Diamond tr(\alpha) & tr(\lhd\alpha) & = & \Diamond\neg tr(\alpha).
\end{array}
$$

One problem is that these translated formulas will not be in the above mentioned normal form for classical modal logics. However, it is not difficult to see that after *normalizing* formulas (that is, pushing negations downwards, using de Morgan's laws and for the modalities, replacing $\neg\Box$ with $\Diamond\neg$, etc.), we do obtain classical Sahlqvist–van Benthem formulas. That is:

> if α and β are a left and a right Sahlqvist formula, then $norm(tr(\alpha)) \to norm(tr(\beta))$ is a Sahlqvist–van Benthem formula.

A formal proof of this fact is in essence straightforward, but technically rather tedious and involved; details are left for the interested reader.

Using existing results, it is in fact not very difficult to prove that any sequent of the form $\alpha \Rightarrow \beta$, with α a left and β a right Sahlqvist formula, is canonical. But we can do somewhat better.

For, notice that in the definition of a Sahlqvist–van Benthem formula $\phi \to \psi$, there is no requirement on the formulas ϕ and ψ being uniform. The only requirement on ϕ is the 'path condition' stating that on no path from a *positive* occurrence of a proposition letter one meets a choice node before a universal one; a similar requirement applies to ψ. Obviously, we may impose the very same conditions on the left- and right-hand formulas of a sequent in distributive modal logic, and we can certainly prove a correspondence and a canonicity result for such sequents. (The reader is invited to check that this requirement identifies the $(1, \ldots, 1)$-Sahlqvist sequents as defined below.)

Our ultimate definition is even more general than this. What is at stake here is a subtle yet crucial difference between distributive and classical modal logic. In the latter case, when it comes to frame validity, or to axiomatizing a logic, any formula may be replaced by any of its substitution instances in which we replace some of the variables uniformly with their *negation*. It may be the case that such a substitution turns a formula that is not in Sahlqvist–van Benthem shape into one that is. This is so much the better since it considerably widens the scope of Sahlqvist theory by providing Sahlqvist–van Benthem equivalents for formulas that are not in the required shape: the idea is to manipulate a given formula until it is obviously in a good Sahlqvist form. In the negation free case such 'preprocessing' is of course not allowed, simply because the required substitutions, crucially involving negation, are not available. However, using the notion of order type, we can instead simply *list* all the possible options.

We are now ready to define general Sahlqvist sequents.

Definition 3.4. Let $\varepsilon \in \{1, \partial\}^n$ be an order type. A formula $\alpha(x_1, \ldots, x_n)$ is *ε-left Sahlqvist* (resp. *ε-right Sahlqvist*) if it satisfies the following two conditions:

(1) in the positive (resp. negative) generation tree, for all i with $\varepsilon_i = 1$, there are no paths from an occurrence of x_i with $+$ to the root along which one meets a choice node before a universal node, and
(2) in the positive (resp. negative) generation tree, for all i with $\varepsilon_i = \partial$, there are no paths from an occurrence of x_i with $-$ to the root along which one meets a choice node before a universal node.

An *ε-Sahlqvist sequent* is a sequent $\alpha \Rightarrow \beta$ such that α is ε-left Sahlqvist and β ε-right Sahlqvist. A sequent is called simply a *Sahlqvist sequent* if it is an ε-Sahlqvist sequent for some order type ε. A distributive modal logic Λ is said to be Sahlqvist provided there is a set Γ of Sahlqvist sequents so that $\Lambda = \mathbf{DM}.\Gamma$.

Using the ideas described earlier on, it is now a tedious but straightforward exercise to show that Sahlqvist sequents restrict to Sahlqvist formulas in the classical setting.

Before we can move on to formulate our main results, there is still a bit of terminology that we have to introduce.

Definition 3.5. Let $\varepsilon \in \{1, \partial\}^n$ be an order type. A formula $\alpha(x_1, \ldots, x_n)$ is *ε-positive (ε-negative)* if in any generation tree of the formula

(1) for all i with $\varepsilon_i = 1$, all the occurrences of x_i have the same (opposite) sign as the initial condition, and
(2) for all i with $\varepsilon_i = \partial$, all the occurrences of x_i have the opposite (same) sign as the initial condition.

Observe that these notions can be seen as providing more detailed information concerning *uniform* formulas, in the sense that every uniform formula is ε-positive for some $\varepsilon \in \{1, \partial\}^n$, and vice versa. In Section 5 we will see that this definition can and will be applied as well to other formulas/terms than those of distributive modal logic.

3.1. Formulation of main results

We are now ready to state the canonicity, correspondence and completeness parts of the Sahlqvist theorem for distributive modal logics.

Theorem 3.6 (Canonicity for Sahlqvist DML). *Every Sahlqvist distributive modal logic is canonical, and hence complete.*

Theorem 3.7 (Correspondence for Sahlqvist DML). *Every Sahlqvist modal sequent corresponds to a formula in the first order language of frames for distributive modal logic. This first order formula can be effectively computed from the modal sequent.*

Combining these two theorems, we obtain the following result.

Theorem 3.8 (Sahlqvist Completeness Theorem for DML). *Every Sahlqvist distributive modal logic* $\mathbf{DM}.\Gamma$ *is sound and complete with respect to the elementary class of frames defined by the (set of) first order correspondents of the axioms* Γ.

We will prove the canonicity for distributive modal logics algebraically: Theorem 5.1 states that the inequalities corresponding to Sahlqvist sequents are canonical. As we saw in Lemma 2.30, completeness follows from canonicity. This takes care of proving Theorem 3.6. The correspondence result, Theorem 3.7, will be proved in Section 4; the proof is based on a reduction to the classical case. Finally, the completeness result, Theorem 3.8, is a simple corollary of the previous two theorems.

4. Correspondence

The main purpose of this section is to prove:

Theorem 3.7 (Correspondence for Sahlqvist DML). *Every Sahlqvist modal sequent corresponds to a formula in the first order language of frames for distributive modal logic. This first order formula can be effectively computed from the modal sequent.*

Proof. It is already known that this holds in the classical setting; see Blackburn, de Rijke and Venema [4] or Sambin and Vaccaro [26] for details.

Our result will be based on a reduction to this classical case; the heart of our proof is a translation from distributive modal logic to classical Boolean modal logic, analogous to the well-known Gödel translation from intuitionistic logic to modal logic, cf. [8]. The semantic intuition behind this translation is simply that by considering the ordering $\leq$ of a distributive modal frame $\mathbb{F}$ as just another binary relation on the domain, we may treat $\mathbb{F}$ as a frame for classical (poly-)modal logic. This idea already has a history in the literature on intuitionistic modal logic; cf. [28] for a recent application and further references.

Special about our approach is that we work with modalities for both the order *and* its converse, and that our translation will be indexed by an order type. That is, as its diamonds, our Boolean modal language will have $\Diamond_{\leq}$, $\Diamond_{\geq}$ and $\Diamond_{\heartsuit}$ for each $\heartsuit \in \{\Diamond, \Box, \rhd, \lhd\}$. These modalities will be interpreted in the obvious way. Now for each order type ε we will give a translation B_ε mapping distributive modal formulas/terms to Boolean modal formulas/terms in the language just described. The definition of B_ε proceeds by a formula induction, the interesting part of which is the base clause:

$$B_\varepsilon(x_i) \; = \begin{cases} \Box_{\geq} x_i & \text{if } \varepsilon_i = 1, \\ \Diamond_{\leq} x_i & \text{if } \varepsilon_i = \partial, \end{cases}$$

while the inductive clauses are completely trivial:

$$\begin{array}{lcl@{\qquad\qquad}lcl} B_\varepsilon(\bot) & = & \bot & B_\varepsilon(\top) & = & \top \\ B_\varepsilon(\alpha \vee \beta) & = & B_\varepsilon(\alpha) \vee B_\varepsilon(\beta) & B_\varepsilon(\alpha \wedge \beta) & = & B_\varepsilon(\alpha) \wedge B_\varepsilon(\beta) \\ B_\varepsilon(\Diamond\alpha) & = & \Diamond_{\Diamond} B_\varepsilon(\alpha) & B_\varepsilon(\Box\alpha) & = & \Box_{\Box} B_\varepsilon(\alpha) \\ B_\varepsilon(\rhd\alpha) & = & \neg\Diamond_{\rhd} B_\varepsilon(\alpha) & B_\varepsilon(\lhd\alpha) & = & \Diamond_{\lhd}\neg B_\varepsilon(\alpha) \end{array}$$

where $\Box_*$ is an abbreviation for $\neg\Diamond_*\neg$.

Two claims will together constitute the proof of the correspondence theorem. First we show that on the level of frame validity, the translation preserves validity:

Claim 1. *Let $\alpha \Rightarrow \beta$ be a* DML *sequent, and let $\mathbb{F}$ be some frame. Then for any order type ε:*

$$\mathbb{F} \Vdash \alpha \Rightarrow \beta \iff \mathbb{F} \Vdash B_\varepsilon(\alpha) \to B_\varepsilon(\beta).$$

For a *proof* of this claim, fix $\mathbb{F}$ and ε. Let V_d and V_b be a distributive (i.e., persistent) and a Boolean valuation on $\mathbb{F}$ that are *ε-associates* in the sense that

$$V_d(x_i) \; = \begin{cases} [\geq]V_b(x_i) & \text{if } \varepsilon_i = 1, \\ \langle\leq\rangle V_b(x_i) & \text{if } \varepsilon_i = \partial \end{cases}$$

for all variables x_i. It follows from a straightforward inductive proof that such valuations satisfy, for every DML formula ϕ, and every point s:

$$\mathbb{F}, V_d, s \Vdash \phi \iff \mathbb{F}, V_b, s \Vdash B_\varepsilon(\phi).$$

Now let V_d be an arbitrary distributive valuation. It is easy to see that both $[\geq]$ and $\langle\leq\rangle$ leave down-sets alone, thus V_d and V_d itself, now seen as a Boolean valuation, are ε-associates. Conversely, if V_b is a Boolean valuation then V_d, defined by

$$V_d(x_i) \; := \begin{cases} [\geq]V_b(x_i) & \text{if } \varepsilon_i = 1, \\ \langle\leq\rangle V_b(x_i) & \text{if } \varepsilon_i = \partial \end{cases}$$

for all variables x_i, is easily seen to be persistent. That is, $V_d(x_i)$ is a down-set for each variable x_i. And it is obviously an ε-associate of V_b. From this and the previous equivalence it is easy to derive that

$$\mathbb{F} \not\Vdash \alpha \Rightarrow \beta \iff \mathbb{F} \not\Vdash B_\varepsilon(\alpha) \to B_\varepsilon(\beta),$$

whence the claim is immediate.

Hence, if we can show that *at least one* of the B_ε-translations preserves Sahlqvistness, we are done.

Claim 2. *A* DMA *term φ is ε-left (ε-right) Sahlqvist iff $B_\varepsilon(\varphi)$ is ε-left (ε-right) Sahlqvist.*

We will prove that the classical modal logic term $B_\varepsilon(\varphi)$ is ε-left (ε-right) Sahlqvist rather than using some classical version of Sahlqvist. However, to do this, we need to say a few words about the classical negation $\neg$. Are there occurrences of $\neg$ that are choice and/or universal nodes? Recall that, for our weakened negations, some occurrences are choice and some are universal. But neither negative nor positive nodes are choice nodes for *both* $\rhd$ and $\lhd$ and since an $\neg$ node may be viewed as either kind of node, no $\neg$ node is a choice node. Similarly no $\neg$ node is universal.

Let $\varphi = \varphi(x_1, \ldots, x_n)$ be a DMA term and let $\varepsilon = (\varepsilon_1, \ldots, \varepsilon_n)$. First observe that the change of each connective by the transformation B_ε does not affect Sahlqvistness. For example, suppose a node $\rhd$ in φ carries a $-$. Then it is a choice node, and its successor node must carry a $+$. Applying B_ε, the node $\rhd$ becomes two nodes: $\neg$ followed by $\Diamond_\rhd$. Since $\neg$ carries a $-$, the node $\Diamond_\rhd$ must carry a $+$, and therefore it is a choice node, and its successor node must carry a $+$. One can check all other cases similarly. Now suppose $\varepsilon_i = 1$, then $B_\varepsilon(x_i) = \Box_{\geq} x_i$ by definition. By the above observation, $\Box_{\geq}$ carries the same sign in a signed generation tree for $B_\varepsilon(\varphi)$ as x_i carries in the signed generation tree for φ with the same initial condition. And because $\Box_{\geq}$ is sign preserving, it follows that x_i carries the same sign in a signed generation tree for $B_\varepsilon(\varphi)$ as x_i carries in the signed generation tree for φ with the same initial condition. If an occurrence of x_i in φ carries a $-$, then Sahlqvistness makes no stipulations on occurrences of choice and universal nodes in the path from this occurrence of x_i to the root of a generation tree of φ. And thus it does not either in the corresponding generation tree for $B_\varepsilon(\varphi)$. On the other hand, if an occurrence x_i in a generation tree for φ carries a $+$, then so does $\Box_{\geq}$ in the corresponding signed generation tree for $B_\varepsilon(\varphi)$. Thus $\Box_{\geq}$ is a universal node. But a universal node just above a variable occurrence does not affect Sahlqvistness. Similarly, if $\varepsilon_i = \partial$, then by definition $B(x_i) = \Diamond_{\leq} x_i$. Here, the interesting case occurs when x_i carries $-$. But then so does $\Diamond_{\leq}$, and it is again a universal node, which does not affect Sahlqvistness. This finishes the proof of Claim 2.

Now, using the two claims, one easily derives the correspondence theorem: a Sahlqvist sequent holds on a frame iff its Boolean translation holds on the frame iff the frame satisfies the first order correspondent of the Boolean translation. $\Box$

Remark 4.1. Notice the difference between the two claims in the proof of Theorem 3.7: whereas the first one holds for *all* order types, the second one in general only is guaranteed for the specific order type with respect to which the sequent is Sahlqvist. For example, $\varphi = \Box x$ is 1-left Sahlqvist but $B_1(\varphi) = \Box\Diamond_{\leq} x$ is not.

Example 4.2. $\top \preccurlyeq x \vee \lhd x$.

Note that this equation is 1-Sahlqvist. Though we can calculate the correspondent by using the reduction to the classical result, it is usually easier to calculate it directly, which we will do here. (We let $R_{\lhd}[u]$ denote the set of points v such that $u R_{\lhd} v$.)

$$
\begin{array}{ll}
 & \mathbb{F}^{+} \models \top \preccurlyeq x \vee \lhd x \\
\text{iff} & \forall S \in \mathcal{D}(W) : W \subseteq S \cup \langle R_{\lhd}] S \\
\text{iff} & \forall S \forall u : u \in W \to u \in S \cup \langle R_{\lhd}] S \\
\text{iff} & \forall S \forall u : u \in W \to u \in S \text{ or } u \in \langle R_{\lhd}] S \\
\text{iff} & \forall u \forall S : u \in - \langle R_{\lhd}] S \to u \in S \\
\text{iff} & \forall u \forall S : u \in [R_{\lhd}] S \to u \in S \\
\text{iff} & \forall u \forall S : R_{\lhd}[u] \subseteq S \to u \in S \\
\text{iff}^{(*)} & \forall u : u \in \quad R_{\lhd}[u] \\
\text{iff} & \forall u : u R_{\lhd} u.
\end{array}
$$

The key is to eliminate the universal quantifier on elements of $\mathcal{D}(W)$. For $(*)$, it suffices to show $(\forall S \in \mathcal{D}(W) : R_{\lhd}[u] \subseteq S \to u \in S)$ iff $u \in \quad R_{\lhd}[u]$ for all $u \in W$. The forward direction follows since $R_{\lhd} \circ \geq \subseteq R_{\lhd}$ implies that $R_{\lhd}[u]$ is a down-set and thus we can choose it as an instance of a down-set containing itself. For the converse, notice that $R_{\lhd}[u]$ is the minimum down-set containing $R_{\lhd}[u]$. So $u \in R_{\lhd}[u]$ gives $u \in S$.

Notice that the correspondent can be expressed in short form as $\Delta \subseteq R_{\lhd}$ where $\Delta = \{(u, u) : u \in W\}$ is the diagonal.

5. Canonicity of Sahlqvist logics

In this section we show that the validity of Sahlqvist inequalities $\alpha \preccurlyeq \beta$ (that is, the inequalities corresponding to Sahlqvist sequents $\alpha \Rightarrow \beta$) is preserved when we move to the canonical extension of a distributive modal algebra. That is, we will prove the following Theorem.

Theorem 5.1. *Every Sahlqvist inequality is canonical for* DMAs.

In order to explain our approach towards canonicity, which is based on and generalizes that taken by Jónsson in [20], suppose that the inequality $\alpha \preccurlyeq \beta$ holds in the algebra $\mathbb{A}$. Recall that with each term $\alpha(x_1, \dots, x_n)$ we may associate a term function $\alpha^{\mathbb{A}} : A^n \to A$, and that $\mathbb{A} \models \alpha \preccurlyeq \beta$ simply means that $\alpha^{\mathbb{A}} \leq \beta^{\mathbb{A}}$ (that is, $\alpha^{\mathbb{A}}(a_1, \dots, a_n) \leq \beta^{\mathbb{A}}(a_1, \dots, a_n)$ for all $a_1, \dots, a_n$ in A). Likewise, $\mathbb{A}^{\sigma} \models \alpha \preccurlyeq \beta$ means that $\alpha^{\mathbb{A}^{\sigma}} \leq \beta^{\mathbb{A}^{\sigma}}$. But from $\alpha^{\mathbb{A}} \leq \beta^{\mathbb{A}}$ it is not difficult to infer that $(\alpha^{\mathbb{A}})^{\sigma} \leq (\beta^{\mathbb{A}})^{\sigma}$, so if we could prove (for every algebra $\mathbb{A}$) that $\alpha^{\mathbb{A}^{\sigma}} \leq (\alpha^{\mathbb{A}})^{\sigma}$ and $(\beta^{\mathbb{A}})^{\sigma} \leq \beta^{\mathbb{A}^{\sigma}}$, we would have established the canonicity of the inequality $\alpha \preccurlyeq \beta$.

This inspires the following definition, which allows us to formulate the above argument concisely as follows: if α is a σ-expanding term, and β is σ-contracting, then the inequality $\alpha \preccurlyeq \beta$ is canonical. This definition was first given explicitly in [20].

Definition 5.2. Let $\alpha(x_1, \dots, x_n)$ be a DMA term, and let $(\cdot)^{\lambda}$ be a uniform way of extending operations on DMAs to operations on their canonical extensions. A DMA term α

is called λ-*expanding* if $\alpha^{\mathbb{A}^\sigma} \leq (\alpha^{\mathbb{A}})^\lambda$, and λ-*contracting* if $(\alpha^{\mathbb{A}})^\lambda \leq \alpha^{\mathbb{A}^\sigma}$. Terms that are both λ-expanding and contracting are called λ-*stable*.

We are now ready for the proof of the algebraic canonicity result.

Proof of Theorem 5.1. The proof is based on a combination of the following ideas, each of which is of interest in its own right, and hence, will be stated as a separate lemma:

***Lemma* 5.14.** Every Sahlqvist inequality $\alpha \preccurlyeq \beta$ can be effectively rewritten into an equivalent inequality of the form $\alpha_1 \preccurlyeq \beta_1 \vee \gamma$, such that for some ε, the term α_1 is an ε-positive left Sahlqvist term, β_1 is an ε-negative right Sahlqvist term, and γ is an ε-positive term.

A similar idea is used in [20]. As we will see, the point is that it will be much easier to work with the terms α_1 and β_1 because they are uniform. The price that we have to pay is the extra term γ, and what complicates matters is that in order to define γ, we have to *extend* the similarity type. That is, augment the type of each DMA by a basic binary operation denoted by the operation symbol n, and given by:

$$n^{\mathbb{A}}(a,b) = \begin{cases} \bot & \text{if } a \leq b \\ \top & \text{if } a \not\leq b. \end{cases} \tag{1}$$

Notice that $n^{\mathbb{A}} : \mathbb{A} \times \mathbb{A}^\partial \to \mathbb{A}$ is an operator, that is, it preserves binary joins in the first coordinate and turns meets into joins in the second coordinate. In particular, $n^{\mathbb{A}}$ is a *monotone* operation—this will be crucial in proving that the new term γ behaves well. Notice that in the classical setting, the classical negation allows one to add just a unary operator; see [20].

In general, the fact that we have expanded the similarity type implies that we also have to formulate and prove some of our lemmas in a slightly more general context.

***Lemma* 5.10.** Every left (right) Sahlqvist DMA term is σ-stable (π-stable).

***Lemma* 5.12.** Let β be an ε-negative, and γ an ε-positive DLM-term, for some order type ε. Then for any DML $\mathbb{A}$ of the right type, we have $(\beta^{\mathbb{A}} \vee \gamma^{\mathbb{A}})^\sigma \leq (\beta^{\mathbb{A}})^\pi \vee (\gamma^{\mathbb{A}})^\sigma$.

***Lemma* 5.5.** Every uniform DLM-term is both σ-contracting and π-expanding. (Hence, in particular, every ε-positive DMA-term is σ-contracting.)

Altogether this allows us to prove the canonicity of Sahlqvist equations in the following way. Consider a Sahlqvist inequality $\alpha \preccurlyeq \beta$ and let α_1, β_1 and γ be the ε-positive left Sahlqvist, the ε-negative right Sahlqvist, and the ε-positive term such that $\alpha \preccurlyeq \beta$ is equivalent to the inequality $\alpha_1 \preccurlyeq \gamma \vee \beta_1$. The theorem is proved by the following sequence of implications and equivalences:

$$\begin{aligned}
& \mathbb{A} \models \alpha \preccurlyeq \beta \\
\overset{5.14}{\iff} \quad & \mathbb{A} \models \alpha_1 \preccurlyeq \beta_1 \vee \gamma \\
\iff \quad & \alpha_1^{\mathbb{A}} \leq (\beta_1 \vee \gamma)^{\mathbb{A}} \\
\iff \quad & \left(\alpha_1^{\mathbb{A}}\right)^\sigma \leq \left((\beta_1 \vee \gamma)^{\mathbb{A}}\right)^\sigma = \left(\beta_1^{\mathbb{A}} \vee \gamma^{\mathbb{A}}\right)^\sigma \\
\overset{5.12}{\Longrightarrow} \quad & \left(\alpha_1^{\mathbb{A}}\right)^\sigma \leq \left(\beta_1^{\mathbb{A}}\right)^\pi \vee \left(\gamma^{\mathbb{A}}\right)^\sigma
\end{aligned}$$

$$\stackrel{5.5}{\Longrightarrow} \quad \left(\alpha_1^{\mathbb{A}}\right)^{\sigma} \leq \left(\beta_1^{\mathbb{A}}\right)^{\pi} \vee \gamma^{\mathbb{A}^\sigma}$$
$$\stackrel{5.10}{\Longrightarrow} \quad \alpha_1^{\mathbb{A}^\sigma} \leq \beta_1^{\mathbb{A}^\sigma} \vee \gamma^{\mathbb{A}^\sigma}$$
$$\Longleftrightarrow \quad \alpha_1^{\mathbb{A}^\sigma} \leq (\beta_1 \vee \gamma)^{\mathbb{A}^\sigma}$$
$$\Longleftrightarrow \quad \mathbb{A}^\sigma \models \alpha_1 \preccurlyeq \beta_1 \vee \gamma$$
$$\stackrel{5.14\&5.15}{\Longleftrightarrow} \quad \mathbb{A}^\sigma \models \alpha \preccurlyeq \beta,$$

where the equivalences without explicit justification follow more or less directly from the definitions. Note that for the validity of the last equivalence we need that our newly defined operation n interacts well with taking canonical extensions; that is, in Lemma 5.15 we will show that $(n^{\mathbb{A}})^\sigma$ is in fact the map $n^{\mathbb{A}^\sigma}$ defined on $\mathbb{A}^\sigma$ as in (1). □

In the remainder of this section, we prove or at least state the technical results needed in this argument.

The basic fact needed to show that uniform terms are σ-contracting, that is, $(\gamma^{\mathbb{A}})^\sigma \leq \gamma^{\mathbb{A}^\sigma}$, dates back to Ribeiro's paper [24].

Theorem 5.3. *For any DL maps $f : \mathbb{A} \to \mathbb{B}$ and $g : \mathbb{B} \to \mathbb{C}$ if f and g are order preserving then $(gf)^\sigma \leq g^\sigma f^\sigma$.*

Proof. Since all maps involved are order preserving, by Remark 2.17 we have that

$$(gf)^\sigma(u) = \bigvee\left\{\bigwedge\{gf(a) : x \leq a \in A\} : u \geq x \in K(\mathbb{A}^\sigma)\right\},$$
$$g^\sigma f^\sigma(u) = \bigvee\left\{\bigwedge\{g(b) : z \leq b \in B\} : f^\sigma(u) \geq z \in K(\mathbb{B}^\sigma)\right\}.$$

Hence, in order to show that $(gf)^\sigma(u) \leq g^\sigma f^\sigma(u)$ it suffices to prove that each of the joinands in the first line is below one of the joinands in the second.

Take an arbitrary $x \in K(\mathbb{A}^\sigma)$ with $x \leq u$. As f^σ is order preserving, we have that $f^\sigma(x) \leq f^\sigma(u)$, and since $f^\sigma(x)$ is closed, it is actually one of those elements $z \leq f^\sigma(u)$ in $K(\mathbb{B}^\sigma)$ that are mentioned in the above characterization of $g^\sigma f^\sigma(u)$.

Hence it suffices to show that $\bigwedge\{gf(a) : x \leq a \in A\}$ in the first formula is below the special joinand $\bigwedge\{g(b) : f^\sigma(x) \leq b \in B\}$ in the second. To show this, let $f^\sigma(x) \leq b \in B$. That is, $\bigwedge\{f(a) : x \leq a \in A\} \leq b$, then by the compactness property of the canonical extension there are $a_1, \ldots, a_n$ with $x \leq a_i \in A$ and $\bigwedge_{i=1}^n f(a_i) \leq b$. But then $a = \bigwedge_{i=1}^n a_i \in A$, $x \leq a$, and $gf(a) = gf(\bigwedge_{i=1}^n a_i) \leq g(\bigwedge_{i=1}^n f(a_i)) \leq g(b)$. Thus for each $b \in B$ with $f^\sigma(x) \leq b$ we have $\bigwedge\{gf(a) : x \leq a \in A\} \leq g(b)$ and therefore $\bigwedge\{gf(a) : x \leq a \in A\} \leq \bigwedge\{g(b) : f^\sigma(x) \leq b \in B\}$. □

For the proper formulation of the next result, we need to generalize some of our terminology concerning DMA terms to the more general context of *monotone* bounded distributive lattice expansions; cf. [14].

Definition 5.4. A *distributive lattice expansion* or DLE is any algebra $\mathbb{A} = (\mathbb{D}, \{f_i\}_{i\in I})$ consisting of a DL $\mathbb{D} = (A, \wedge, \vee, \bot, \top)$ and additional operations $f_i : A^{n_i} \to A$ for each $i \in I$. Such a DLE is said to be *monotone*, and is then called a DLM, provided each basic operation is monotone, that is, for each $i \in I$, there is an order type $\varepsilon_i \in \{1, \partial\}^{n_i}$ so that $f_i : A^{\varepsilon_i} \to A$ is order preserving. The sequence $(\varepsilon_i)_{i\in I}$ is called the monotonicity type of $\mathbb{A}$.

Of course, DMAs are special DLMs, and so is any DMA expanded with the operation n given in (1).

For DLM terms of some monotonicity type $\tau = (\varepsilon_i)_{i\in I}$ we may also define signed generation trees as we have done for DMA terms. Only here, if $\varepsilon_i(j) = 1$, $1 \le j \le n_i$, then the j-th child of a node labelled f_i will be given the same sign as the node itself, and if $\varepsilon_i(j) = \partial$, $1 \le j \le n_i$, then the j-th child of the node labelled f_i will be given the opposite sign from the one the node itself carries. Uniform terms as well as λ-contracting and λ-expanding terms are all defined the same way for DLMs of a given monotonicity type as we defined them for DMAs.

We formulate and prove the lemma in this setting.

Lemma 5.5. *Every uniform* DLM*-term is both σ-contracting and π-expanding.*

Proof. We need to prove $(\gamma^{\mathbb{A}})^\sigma \le \gamma^{\mathbb{A}^\sigma}$ and $(\gamma^{\mathbb{A}})^\pi \ge \gamma^{\mathbb{A}^\sigma}$ for a uniform term γ and a DLM $\mathbb{A}$. We prove it by induction on the complexity of γ. When γ is a constant or a variable, the conclusion is easily verified as projections extend to the corresponding projections. As an inductive step, we consider only the case $\gamma = f(\gamma_1, \gamma_2)$ where $f^{\mathbb{A}}$ is order preserving in the first coordinate and order reversing in the second.

Let $x_1, \ldots, x_n$ be a set of variables containing the ones occurring in γ. The fact that γ is uniform implies that there is an $\varepsilon \in \{1, \partial\}^n$ so that γ is ε-positive. But now since $f^{\mathbb{A}}$ is order preserving in the first coordinate and order reversing in the second, it follows by the way we assign signs to nodes that the subterms γ_1 and γ_2 are ε-positive and ε-negative, respectively. That is, we have that $\gamma_1^{\mathbb{A}} : A^\varepsilon \to A$ is order preserving, $\gamma_2^{\mathbb{A}} : A^\varepsilon \to A$ is order reversing, and thus the order variant $(\gamma_2^{\mathbb{A}})^\partial : A^\varepsilon \to A^\partial$ of $\gamma_2^{\mathbb{A}}$ is order preserving. So $\gamma^{\mathbb{A}} : A^\varepsilon \to A$ is given by the composition

$$A^\varepsilon \xrightarrow{\left(\gamma_1^{\mathbb{A}}, (\gamma_2^{\mathbb{A}})^\partial\right)} A \times A^\partial \xrightarrow{f^{\mathbb{A}}} A$$

of order preserving maps. Thus by Theorem 5.3 it follows that

$$\left(\gamma^{\mathbb{A}}\right)^\sigma \le \left(f^{\mathbb{A}}\right)^\sigma \left(\gamma_1^{\mathbb{A}}, \left(\gamma_2^{\mathbb{A}}\right)^\partial\right)^\sigma$$

and it is a straightforward verification that

$$\left(\gamma_1^{\mathbb{A}}, \left(\gamma_2^{\mathbb{A}}\right)^\partial\right)^\sigma = \left(\left(\gamma_1^{\mathbb{A}}\right)^\sigma, \left(\left(\gamma_2^{\mathbb{A}}\right)^\partial\right)^\sigma\right).$$

As we saw in Remark 2.16

$$\left(\left(\gamma_2^{\mathbb{A}}\right)^\partial\right)^\sigma = \left(\left(\gamma_2^{\mathbb{A}}\right)^\pi\right)^\partial,$$

and we may use the inductive hypothesis to obtain that $(\gamma_1^{\mathbb{A}})^\sigma \le \gamma_1^{\mathbb{A}^\sigma}$ and $(\gamma_2^{\mathbb{A}})^\pi \ge \gamma_2^{\mathbb{A}^\sigma}$. Note that the latter inequality is the same as $((\gamma_2^{\mathbb{A}})^\pi)^\partial \le (\gamma_2^{\mathbb{A}^\sigma})^\partial$. This means that as maps from A^ε to $A \times A^\partial$

$$\left(\left(\gamma_1^{\mathbb{A}}\right)^\sigma, \left(\left(\gamma_2^{\mathbb{A}}\right)^\partial\right)^\sigma\right) \le \left(\gamma_1^{\mathbb{A}^\sigma}, \left(\gamma_2^{\mathbb{A}^\sigma}\right)^\partial\right).$$

Putting all of this together we get

$$\begin{aligned}\left(\gamma^{\mathbb{A}}\right)^{\sigma} &\leq \left(f^{\mathbb{A}}\right)^{\sigma}\left(\gamma_1^{\mathbb{A}^{\sigma}}, \left(\gamma_2^{\mathbb{A}^{\sigma}}\right)^{\partial}\right)\\ &= f^{\mathbb{A}^{\sigma}}\left(\gamma_1^{\mathbb{A}^{\sigma}}, \left(\gamma_2^{\mathbb{A}^{\sigma}}\right)^{\partial}\right)\\ &= \gamma^{\mathbb{A}^{\sigma}}\end{aligned}$$

which is exactly what we desired. The fact that γ is π-contracting is proved dually with the additional step

$$\left(\gamma^{\mathbb{A}}\right)^{\pi} \geq \left(f^{\mathbb{A}}\right)^{\pi}\left(\gamma_1^{\mathbb{A}}, \left(\gamma_2^{\mathbb{A}}\right)^{\partial}\right)^{\pi} \geq \left(f^{\mathbb{A}}\right)^{\sigma}\left(\gamma_1^{\mathbb{A}}, \left(\gamma_2^{\mathbb{A}}\right)^{\partial}\right)^{\pi},$$

which holds since the π-extension of any map is greater than the σ-extension of that map. □

The two fundamental technical results needed to show that left and right Sahlqvist terms are stable with respect to σ- and π-extensions, respectively, date back to the original Jónsson–Tarski paper [21] and to the Gehrke–Jónsson paper [14], respectively. Since left and right Sahlqvist terms are uniform by definition, they are σ-contracting and π-expanding by Lemma 5.5; recall that in the proof of this lemma, we needed Ribeiro's Theorem 5.3 stating that for order preserving DL maps $f : \mathbb{A} \to \mathbb{B}$ and $g : \mathbb{B} \to \mathbb{C}$ we have that

$$(gf)^{\sigma} \leq g^{\sigma} f^{\sigma}.$$

In order to show that left and right Sahlqvist terms are also σ-contracting and π-expanding, we will need to show the converse inequality. In the Corollaries 5.7 and 5.9 below we will see that, under additional constraints on f and/or g, we can indeed prove that

$$g^{\sigma} f^{\sigma} \leq (gf)^{\sigma},$$

establishing the (conditional) *functoriality* of $(\cdot)^{\sigma}$. In the case of both corollaries this functoriality is proved by showing that the 'additional constraints' on a map h make that its extensions h^{σ} and h^{π} will satisfy some strong continuity principles. The difference between the corollaries is that in the one case the 'additional constraints' are imposed on f (the 'first' map), and in the other case on g (the 'second' map).

The first result states that the canonical extension of an *operator* is Scott continuous. (Recall that $h : \mathbb{B}^n \to \mathbb{C}$ is an operator if it preserves joins in each coordinate.)

Theorem 5.6. *If the DL map $g : \mathbb{B}^n \to \mathbb{C}$ is an operator then g^{σ} is Scott continuous, i.e.,*

(Scott) *For all $u \in B^{\sigma n}$ and for all $q \in J^{\infty}(\mathbb{C}^{\sigma})$, if $q \leq g^{\sigma}(u)$ then there exists $p \in J^{\infty}(\mathbb{B}^{\sigma n})$ so that $p \leq u$ and $q \leq g^{\sigma}(v)$ for all $v \in B^{\sigma n}$ with $p \leq v$.*

Proof. See [13, Lemma 4.2, p. 213]. □

The first corollary then is about the composition of two maps of which the *second* map is nice:

Corollary 5.7. *If $f : \mathbb{A} \to \mathbb{B}^n$ is any map between DLs and $g : \mathbb{B}^n \to \mathbb{C}$ is an operator then $g^\sigma f^\sigma \leq (gf)^\sigma$.*

Proof. It is straightforward to check that since f^σ satisfies (UC) and since g^σ satisfies (Scott), it follows that $g^\sigma f^\sigma$ satisfies (UC). Now since $(gf)^\sigma$ is the greatest extension of gf which satisfies (UC), the result follows. □

In the second case we look at maps that are *meet preserving*. Note that, in the case of unary operations, this is the same concept as being a dual operator; for operations of higher rank, however, preserving meets is a far stronger condition.

Theorem 5.8. *If the DL map $f : \mathbb{A} \to \mathbb{B}$ is meet preserving then f^σ is 'strongly upper continuous', i.e.,*

(SUC) *For all $u \in A^\sigma$ and for all $z \in K(\mathbb{B}^\sigma)$, if $z \leq f^\sigma(u)$ then there exist $x \in K(\mathbb{A}^\sigma)$ so that $x \leq u$ and $z \leq f^\sigma(v)$ for all $v \in A^\sigma$ with $x \leq v$.*

Proof. See [15, Theorem 2.27]. □

The second corollary is about the composition of two maps of which the *first* maps is very nice (and of which the second map is order preserving):

Corollary 5.9. *If $f : \mathbb{A} \to \mathbb{B}$ is meet preserving and $g : \mathbb{B} \to \mathbb{C}$ is order preserving, then $g^\sigma f^\sigma \leq (gf)^\sigma$.*

Proof. It is straightforward to check that since f^σ satisfies (SUC) and since g^σ satisfies the one sided version of (UC) which holds for order preserving maps, it follows that $g^\sigma f^\sigma$ satisfies (UC). Now since $(gf)^\sigma$ is the greatest extension of gf which satisfies (UC), the result follows. □

Lemma 5.10. *Every left (right) Sahlqvist* DMA *term is σ-stable (π-stable).*

Proof. The fact that left (right) Sahlqvist terms are σ-contracting (π-expanding) follows from Lemma 5.5. We prove the remaining part of the stability by induction. When α is a constant or a variable, the conclusion is easily seen to hold. For the inductive step, we prove $(\alpha^{\mathbb{A}})^\sigma = \alpha^{\mathbb{A}^\sigma}$ for a left Sahlqvist term α. The proof for right Sahlqvist terms is dual.

So suppose that $\alpha = f(\alpha_1, \ldots, \alpha_n)$, $n = 1$ or 2, and that the lemma holds for each of the terms α_i.

The cases in which f is $\vee$, $\wedge$, and $\Diamond$ are similar as f is an operator in all these cases. We do the case where f is binary. It is clear from the definition of left Sahlqvist that if $\alpha = f(\alpha_1, \alpha_2)$ is left Sahlqvist then so are α_1 and α_2. Now given a DMA $\mathbb{A}$, $f^{\mathbb{A}}$ is an operator, and $\alpha^{\mathbb{A}} = (f(\alpha_1, \alpha_2))^{\mathbb{A}} = f^{\mathbb{A}}(\alpha_1^{\mathbb{A}}, \alpha_2^{\mathbb{A}})$, so by Corollary 5.7

$$\left(\alpha^{\mathbb{A}}\right)^\sigma = \left(f^{\mathbb{A}}(\alpha_1^{\mathbb{A}}, \alpha_2^{\mathbb{A}})\right)^\sigma \geq \left(f^{\mathbb{A}}\right)^\sigma \left((\alpha_1^{\mathbb{A}}, \alpha_2^{\mathbb{A}})\right)^\sigma.$$

But $(f^{\mathbb{A}})^\sigma = f^{\mathbb{A}^\sigma}$ and $\left((\alpha_1^{\mathbb{A}}, \alpha_2^{\mathbb{A}})\right)^\sigma = (\left(\alpha_1^{\mathbb{A}}\right)^\sigma, \left(\alpha_2^{\mathbb{A}}\right)^\sigma)$ so the result follows by the induction hypothesis.

Now if $f = \lhd$, then the negative generation tree for α_1 shows up as the subtree of the positive generation tree for α; using this and the definition of left and right Sahlqvist terms the reader can easily verify that α_1 is right Sahlqvist. Now consider $(\lhd\alpha_1)^{\mathbb{A}} = \lhd^{\mathbb{A}}\alpha_1^{\mathbb{A}}$. Recall that left and right Sahlqvist terms are uniform by definition, so there is an $\varepsilon \in \{1, \partial\}^n$ so that α is ε-positive, which implies that α_1 is ε-negative and we can write $(\lhd\alpha_1)^{\mathbb{A}}$ as the composition

$$A^{\varepsilon} \xrightarrow{(\alpha_1^{\mathbb{A}})^{\partial}} A^{\partial} \xrightarrow{\lhd^{\mathbb{A}}} A.$$

This is again an order preserving function followed by an operator and the rest of the argument is similar to our first case, except that we must use the fact that α_1 is π-expanding by the induction hypothesis and that

$$\left(\left(\alpha_1^{\mathbb{A}}\right)^{\partial}\right)^{\sigma} = \left(\left(\alpha_1^{\mathbb{A}}\right)^{\pi}\right)^{\partial}.$$

Now if $f = \Box$, then the very first node is a universal node, and thus if α is left Sahlqvist it follows that there are no choice nodes anywhere in the signed generation tree for α or in other words α is what we called a universal term. Analyzing the definition of choice nodes one may see that having no such means that the term may be viewed as a composition of meet-preserving maps (with the appropriate flips of coordinates and functions administered). We leave the inductive proof of this to the reader.

Using this fact we see that $\alpha_1^{\mathbb{A}} : A^{\varepsilon} \to A$ is a meet preserving map and thus by Corollary 5.9 we have $(\alpha^{\mathbb{A}})^{\sigma} = (\Box^{\mathbb{A}}\alpha_1^{\mathbb{A}})^{\sigma} \geq (\Box^{\mathbb{A}})^{\sigma}(\alpha_1^{\mathbb{A}})^{\sigma}$ and then $(\Box^{\mathbb{A}})^{\sigma}(\alpha_1^{\mathbb{A}})^{\sigma} \geq \Box^{\mathbb{A}^{\sigma}}\alpha_1^{\mathbb{A}^{\sigma}}$ by the induction hypothesis.

Finally, the case where $f = \rhd$ yields α_1 right universal instead of left universal, and this case is handled as the previous one with the same twist as in the case $f = \lhd$. We leave the details to the reader. $\Box$

The next lemma, which deals with the σ-extension of the join of order preserving and order reversing maps, is brand new.

Lemma 5.11. *If $f, g : \mathbb{A} \to \mathbb{B}$ are maps between DLs and f is order preserving and g is order reversing then $(f \vee g)^{\sigma} \leq f^{\sigma} \vee g^{\pi}$.*

Proof. In order to arrive at a contradiction, suppose that $(f \vee g)^{\sigma} \not\leq f^{\sigma} \vee g^{\pi}$, then for some $u \in A^{\sigma}$, $(f \vee g)^{\sigma}(u) \not\leq f^{\sigma}(u) \vee g^{\pi}(u)$. Since $\mathbb{A}^{\sigma}$ is join generated by the set $J^{\infty}(\mathbb{A}^{\sigma})$ of all completely join irreducible elements of $\mathbb{A}^{\sigma}$, this means that there is some $p \in J^{\infty}(\mathbb{A}^{\sigma})$ such that $p \leq (f \vee g)^{\sigma}(u)$, while on the other hand $p \not\leq f^{\sigma}(u)$ and $p \not\leq g^{\pi}(u)$. Now by definition of σ-extensions, we have

$$\begin{aligned}(f \vee g)^{\sigma}(u) \ = \ \bigvee\Big\{\bigwedge\{f(a) \vee g(a) : a \in [x, y]_A\} : K(\mathbb{A}^{\sigma}) \\ \ni x \leq u \leq y \in O(\mathbb{A}^{\sigma})\Big\}\end{aligned}$$

where we take $[x, y]_S$ for a subset $S \subseteq A^{\sigma}$ to mean $\{s \in S : x \leq s \leq y\}$. Then by the complete join irreducibility of p, $p \leq (f \vee g)^{\sigma}(u)$ implies that p must be below one of the joinands; that is, there must be an interval $[x_0, y_0]$ with x_0 closed and y_0 open, such that

$x_0 \leq u \leq y_0$ and $p \leq \bigwedge\{f(a) \vee g(a) : a \in [x, y]_A\}$. As a consequence,

$$p \leq f(a) \vee g(a) \text{ for each } a \in [x_0, y_0].$$

On the other hand, turning to $f^\sigma(u)$ and $g^\pi(u)$, we leave it for the reader to verify that

$$f^\sigma(u) = \bigvee\{f^\sigma(x) : u \geq x \in K(\mathbb{A}^\sigma)\}$$
$$g^\pi(u) = \bigwedge\{g^\pi(x) : u \geq x \in K(\mathbb{A}^\sigma)\}$$

(cf. Remark 2.17). We claim that

$$\bigvee\{f^\sigma(x) : u \geq x \in K(\mathbb{A}^\sigma)\} = \bigvee\{f^\sigma(x) : x \in [x_0, u]_{K(\mathbb{A}^\sigma)}\},$$
$$\bigwedge\{g^\pi(x) : u \geq x \in K(\mathbb{A}^\sigma)\} = \bigwedge\{g^\pi(x) : x \in [x_0, u]_{K(\mathbb{A}^\sigma)}\}.$$

For the first identity, it is obvious that the right-hand side is below the left-hand side, since on the left we take the join of more elements. For the other direction, take an arbitrary joinand to the left, say $f^\sigma(x)$ with $x \in K(\mathbb{A}^\sigma)$ below u. Now consider the element $x \vee x_0$; this object clearly belongs to $[x_0, u]_{K(\mathbb{A}^\sigma)}$, whence $f^\sigma(x \vee x_0)$ is one of the joinands to the right. But since f^σ is order preserving, we see that $f^\sigma(x) \leq f^\sigma(x \vee x_0)$. This shows that each of the left joinands is below some of the joinands to the right, and thus proves that the join to the left is below the join to the right. Thus we have established the first of the above two identities; we leave the second one to the reader.

From the characterization $g^\pi(u) = \bigwedge\{g^\pi(x) : x \in [x_0, u]_{K(\mathbb{A}^\sigma)}\}$ and the fact that $p \not\leq g^\pi(u)$ we may conclude that there is a *closed* element x_1 between x_0 and y_0 such that $p \not\leq g^\pi(x_1)$. Also, from $f^\sigma(u) = \bigvee\{f^\sigma(x) : x \in [x_0, u]_{K(\mathbb{A}^\sigma)}\}$ we may infer that $p \not\leq f^\sigma(x)$ for *each* $x \in [x_0, u]_{K(\mathbb{A}^\sigma)}$; in particular: $p \not\leq f^\sigma(x_1)$. Since f is order preserving this means

$$\begin{aligned} p \not\leq f^\sigma(x_1) &= \bigwedge\{f(a) : x_1 \leq a \in A\} \\ &= \bigwedge\{f(a) : a \in [x_1, y_0]_A\}. \end{aligned}$$

The second meet is small enough because, by compactness, for each $a \in A$ with $x_1 \leq a$ there is an $a' \in A$ with $x_1 \leq a' \leq a$ and $a' \leq y_0$. Thus there is an element $a_0 \in A$ with $x_1 \leq a_0 \leq y_0$ so that $p \not\leq f(a_0)$. But we also have that $p \not\leq g^\pi(x_1)$, and since g is order reversing this means

$$p \not\leq g^\pi(x_1) = \bigvee\{g(a) : x_1 \leq a \in A\}.$$

That is, for every $a \in A$ with $x_1 \leq a$ we have $p \not\leq g(a)$ and in particular $p \not\leq g(a_0)$. But this means $p \not\leq f(a_0) \vee g(a_0)$ which contradicts our earlier claim. □

In the proof of Theorem 5.1 we need the following corollary of this lemma.

Lemma 5.12. *Let β be an ε-negative, and γ an ε-positive* DLM*-term, for some order type ε. Then for any* DML $\mathbb{A}$ *of the right type, we have*

$$\left(\beta^{\mathbb{A}} \vee \gamma^{\mathbb{A}}\right)^\sigma \leq \left(\beta^{\mathbb{A}}\right)^\pi \vee \left(\gamma^{\mathbb{A}}\right)^\sigma.$$

Proof. By Remark 2.16 we may take $\beta^{\mathbb{A}}$ and $\gamma^{\mathbb{A}}$ to be maps from $\mathbb{A}^{\varepsilon}$ to $\mathbb{A}$. But it is straightforward to verify that in this light, $\beta^{\mathbb{A}}$ and $\gamma^{\mathbb{A}}$ satisfy the conditions of Lemma 5.11. And therefore the result is immediate. □

Now we need to construct, from a Sahlqvist inequality $\alpha \preccurlyeq \beta$, the left Sahlqvist term α', the right Sahlqvist term β', and the uniform term γ so that $\alpha \preccurlyeq \beta$ is equivalent to $\alpha' \preccurlyeq \gamma \vee \beta$. Recall that in order to do this we first augment the type of each DMA by the basic binary operation denoted by the operation symbol n, and given by:

$$n^{\mathbb{A}}(a, b) = \begin{cases} \bot & \text{if } a \leq b \\ \top & \text{if } a \nleq b. \end{cases}$$

The following lemma provides the key tool for the rewriting process of an arbitrary Sahlqvist inequality into an inequality of the shape required in the canonicity proof.

Lemma 5.13. *Let α and β be DML terms and s a specific occurrence of a subterm of α. Let $\alpha' = \alpha(z/s)$ be the term obtained when replacing s in α by the variable z, where z is assumed not to occur in α. Then for any DML $\mathbb{A}$ we have*

(1) *If s is signed '+' in the positive generation tree for α then*

$$\alpha \preccurlyeq \beta \text{ holds in } \mathbb{A} \iff \alpha' \preccurlyeq n(z, s) \vee \beta \text{ holds in } \mathbb{A}.$$

(2) *If s is signed '−' in the positive generation tree for α then*

$$\alpha \preccurlyeq \beta \text{ holds in } \mathbb{A} \iff \alpha' \preccurlyeq n(s, z) \vee \beta \text{ holds in } \mathbb{A}.$$

(3) *If s is signed '+' in the negative generation tree for α then*

$$\beta \preccurlyeq \alpha \text{ holds in } \mathbb{A} \iff \beta \preccurlyeq n(z, s) \vee \alpha' \text{ holds in } \mathbb{A}.$$

(4) *If s is signed '−' in the negative generation tree for α then*

$$\beta \preccurlyeq \alpha \text{ holds in } \mathbb{A} \iff \beta \preccurlyeq n(s, z) \vee \alpha' \text{ holds in } \mathbb{A}.$$

Proof. We prove only the first statement, since the others are similar. First suppose $\alpha' \preccurlyeq n(z, s) \vee \beta$ holds in $\mathbb{A}$. Then for $a_1, \ldots, a_n \in A$ we have

$$\begin{aligned} \alpha^{\mathbb{A}}(a_1, \ldots, a_n) &= \alpha'^{\mathbb{A}}(a_1, \ldots, a_n, s^{\mathbb{A}}(a_1, \ldots, a_n)) \\ &\leq n^{\mathbb{A}}(s^{\mathbb{A}}(a_1, \ldots, a_n), s^{\mathbb{A}}(a_1, \ldots, a_n)) \vee \beta^{\mathbb{A}}(a_1, \ldots, a_n) \\ &= \beta^{\mathbb{A}}(a_1, \ldots, a_n) \end{aligned}$$

and $\alpha \preccurlyeq \beta$ holds in $\mathbb{A}$.

Conversely, if $\alpha \preccurlyeq \beta$ holds in $\mathbb{A}$, for $a_1, \ldots, a_n$ and $a_{n+1} \in A$, we distinguish two cases. First, if $a_{n+1} \nleq s^{\mathbb{A}}(a_1, \ldots, a_n)$ then $n^{\mathbb{A}}(a_{n+1}, s^{\mathbb{A}}(a_1, \ldots, a_n)) = \top$ and $\alpha' \preccurlyeq n(z, s) \vee \beta$ holds at $a_1, \ldots, a_n, a_{n+1}$. Now suppose that $a_{n+1} \leq s^{\mathbb{A}}(a_1, \ldots, a_n)$. Since the root of s bears the sign $+$ in the positive generation tree for α (that is, z is assigned $+$ in the positive generation tree for α'), it must be the case that $\alpha'^{\mathbb{A}}$ is order preserving in z. So from $a_{n+1} \leq s^{\mathbb{A}}(a_1, \ldots, a_n)$ we may infer that

$$\begin{aligned}\alpha'^{\mathbb{A}}(a_1,\ldots,a_n,a_{n+1}) &\leq \alpha'^{\mathbb{A}}(a_1,\ldots,a_n,s^{\mathbb{A}}(a_1,\ldots,a_n)) \\ &= \alpha^{\mathbb{A}}(a_1,\ldots,a_n) \\ &\leq \beta^{\mathbb{A}}(a_1,\ldots,a_n) \\ &= \bot \vee \beta^{\mathbb{A}}(a_1,\ldots,a_n)\end{aligned}$$

and $\alpha' \preccurlyeq n(z,s) \vee \beta$ holds in $\mathbb{A}$. $\square$

Thus we have:

Lemma 5.14. *Every Sahlqvist inequality $\alpha \preccurlyeq \beta$ can be effectively rewritten into an equivalent inequality of the form $\alpha_1 \preccurlyeq \beta_1 \vee \gamma$, such that for some ε, the term α_1 is an ε-positive left Sahlqvist term, β_1 is an ε-negative right Sahlqvist term, and γ is an ε-positive term.*

Proof. Note that the slightly curious wording of the previous lemma makes it fairly easy to see that any Sahlqvist inequality $\alpha \preccurlyeq \beta$ can be rewritten step by step into the required form. In each step of this process, either α or β will be simplified, at the expense of extra terms $n(s,z)$ or $n(z,s)$ turning up on the right-hand side of the inequality. It should be noted that in order to rewrite the inequality $\alpha \preccurlyeq \beta$ we have to *extend* the similarity type. Now for the details.

So let $\alpha \preccurlyeq \beta$ be a Sahlqvist inequality and let $x_1,\ldots,x_n$ be the variables occurring in $\alpha \preccurlyeq \beta$. Then there is an $\varepsilon \in \{1,\partial\}^n$ so that α is ε-left Sahlqvist and β is ε-right Sahlqvist. Let $s_1,\ldots,s_m$ be the maximal ε-negative subterms of α whose roots bear the sign $+$ in the positive generation tree for α. Let $t_1,\ldots,t_l$ be the maximal ε-positive subterms of α whose roots bear the sign $-$ in the negative generation tree for α. Also for β let $s'_1,\ldots,s'_{m'}$ be the maximal ε-positive subterms of β whose roots bear the sign $-$ in the negative generation tree for β, and let $t'_1,\ldots,t'_{l'}$ be the maximal ε-negative subterms of β whose roots bear the sign $+$ in the negative generation tree for β.

Introduce new variables $z_1,\ldots,z_m$, $w_1,\ldots,w_l$, $z'_1,\ldots,z'_{m'}$, and $w'_1,\ldots,w'_{l'}$, all distinct from each other and from each of $x_1,\ldots,x_n$. Let $\overline{\varepsilon} \in \{1,\partial\}^{n+m+l+m'+l'}$ be given by

$$\overline{\varepsilon}_i = \begin{cases} \varepsilon_i & \text{if } 1 \leq i \leq n & (x\text{-variables}) \\ 1 & \text{if } n+1 \leq i \leq n+m & (z\text{-variables}) \\ \partial & \text{if } n+m+1 \leq i \leq n+m+l & (w\text{-variables}) \\ \partial & \text{if } n+m+l+1 \leq i \leq n+m+l+m' & (z'\text{-variables}) \\ 1 & \text{if } n+m+l+m'+1 \leq i \leq n+m+l+m'+l' & (w'\text{-variables}). \end{cases}$$

Then $\alpha_1 = \alpha(z_1/s_1,\ldots,z_m/s_m,w_1/t_1,\ldots,w_l/t_l)$ is $\overline{\varepsilon}$-positive and left Sahlqvist, whereas $\beta_1 = \beta(z'_1/s'_1,\ldots,z'_{m'}/s'_{m'},w'_1/t'_1,\ldots,w'_{l'}/t'_{l'})$ is $\overline{\varepsilon}$-negative and right Sahlqvist. By successive applications of the previous lemma, it can be shown that $\alpha \preccurlyeq \beta$ is equivalent to the following inequality:

$$\begin{aligned}&\alpha(z_1/s_1,\ldots,z_m/s_m,w_1/t_1,\ldots,w_l/t_l) \\ \preccurlyeq\ &\beta(z'_1/s'_1,\ldots,z'_{m'}/s'_{m'},w'_1/t'_1,\ldots,w'_{l'}/t'_{l'}) \\ &\vee \bigvee_{j=1}^{m} n(z_j,s_j) \vee \bigvee_{j=1}^{l} n(t_j,w_j) \vee \bigvee_{j=1}^{m'} n(s'_j,z'_j) \vee \bigvee_{j=1}^{l'} n(w'_j,t'_j).\end{aligned}$$

Here

$$\gamma = \bigvee_{j=1}^{m} n(z_j, s_j) \vee \bigvee_{j=1}^{l} n(t_j, w_j) \vee \bigvee_{j=1}^{m'} n(s'_j, z'_j) \vee \bigvee_{j=1}^{l'} n(w'_j, t'_j)$$

is $\overline{\varepsilon}$-positive and we have rewritten the inequality as desired. □

Notice that Lemma 5.5 readily applies to the term γ that we have obtained: thus γ is σ-contracting. Likewise, it follows from Lemma 5.11 that $(\beta^{\mathbb{A}} \vee \gamma^{\mathbb{A}})^\sigma \leq (\beta^{\mathbb{A}})^\pi \vee (\gamma^{\mathbb{A}})^\sigma$. The final result we need is that $(n^{\mathbb{A}})^\sigma = n^{\mathbb{A}^\sigma}$ for each DL $\mathbb{A}$.

Lemma 5.15. *Let $\mathbb{A}$ be a DL. For $u, v \in A^\sigma$ we have*

$$\left(n^{\mathbb{A}}\right)^\sigma (u, v) = \begin{cases} \bot & \text{if } u \leq v \\ \top & \text{if } u \not\leq v. \end{cases}$$

Proof. Notice that since $n^{\mathbb{A}}$ is order preserving in the first coordinate and order reversing in the second coordinate it follows that

$$\left(n^{\mathbb{A}}\right)^\sigma (u, v) = \bigvee \left\{ \left(n^{\mathbb{A}}\right)^\sigma (x, y) : u \geq x \in K(\mathbf{A}^\sigma) \text{ and } v \leq y \in O(\mathbf{A}^\sigma) \right\},$$

and for $x \in K(\mathbf{A}^\sigma)$ and $y \in O(\mathbf{A}^\sigma)$

$$\left(n^{\mathbb{A}}\right)^\sigma (x, y) = \bigwedge \{n^{\mathbb{A}}(a, b) : x \leq a \in A \text{ and } y \geq b \in A\}.$$

Now let $u, v \in A^\sigma$ with $u \leq v$. Then for each $x \in K(\mathbf{A}^\sigma)$ with $x \leq u$ and for each $y \in O(\mathbf{A}^\sigma)$ with $v \leq y$, we have $x \leq u \leq v \leq y$. Thus

$$\bigwedge \{a \in A : a \geq x\} = x \leq y = \bigvee \{a \in A : a \leq y\}$$

and by the compactness property of A^σ it follows that there is $a_0 \in A$ with $x \leq a_0 \leq y$. But then $(n^{\mathbb{A}})^\sigma (x, y) \leq n^{\mathbb{A}}(a_0, a_0) = \bot$. Thus for each $x \in K(\mathbf{A}^\sigma)$ with $x \leq u$ and for each $y \in O(\mathbf{A}^\sigma)$ with $v \leq y$, we have $(n^{\mathbb{A}})^\sigma (x, y) = \bot$ and it follows that $(n^{\mathbb{A}})^\sigma (u, v) = \bot$ as desired.

On the other hand, if $(n^{\mathbb{A}})^\sigma (u, v) = \bot$ then for each $x \in K(\mathbf{A}^\sigma)$ with $x \leq u$ and for each $y \in O(\mathbf{A}^\sigma)$ with $v \leq y$, there are $a, b \in A$ with $x \leq a$ and $b \leq y$ and $n^{\mathbb{A}}(a, b) = \bot$. But then $x \leq a \leq b \leq y$ and thus

$$u = \bigvee \{x \in K(\mathbf{A}^\sigma) : x \leq u\} \leq \bigwedge \{y \in O(\mathbf{A}^\sigma) : v \leq y\} = v.$$

Finally, it is clear from the definition that $(n^{\mathbb{A}})^\sigma (u, v)$ must be either $\bot$ or $\top$ and the claim follows. □

We have now proved all the lemmas that were needed in the proof of Theorem 5.1.

6. Examples

6.1. Positive modal logic

As we mentioned already, Positive Modal Logic (PML) was introduced by Dunn in [11]. In our terminology, a positive modal logic is a DML with a $\Diamond$ and a $\Box$ satisfying the so-called *interaction* axioms

$$\begin{aligned} \Diamond x \wedge \Box y &\Rightarrow \Diamond(x \wedge y) \\ \Box(x \vee y) &\Rightarrow \Box x \vee \Diamond y. \end{aligned}$$

Dunn also showed that PML is complete with respect to classical Kripke frames in which $R_\Diamond = R_\Box$. However, in order to get reasonable results on frame completeness for extensions of PML, ordered Kripke frames were introduced by Celani and Jansana; cf. [6]. They defined a Kripke frame for positive modal logic (let us call it a *P-frame* for now) to be a triple $\mathbb{G} = (W, \leq, R)$ where $(W, \leq)$ is a quasi-ordered set and R is a binary relation on W satisfying

$$\begin{aligned} \leq \circ R &\subseteq R \circ \leq \\ \geq \circ R &\subseteq R \circ \geq. \end{aligned} \tag{P}$$

They associate with such a frame the complex algebra $\mathbb{G}^+ = (\mathcal{D}(W), \langle R\rangle, [R])$[1] where $\langle R\rangle$, $[R]$, and $\mathcal{D}(W)$ are defined as we have done it here. Also, satisfaction is defined such that $\mathbb{G} \Vdash \alpha \Rightarrow \beta$ if and only if $\mathbb{G}^+ \Vdash \alpha \preccurlyeq \beta$ as we have done here. The fact that they consider quasi-orders rather than partial orders makes no major difference since we may factor the equivalence relation corresponding to the quasi-order out of W, $\leq$, and R and get a partially ordered frame with isomorphic complex algebra, and thus satisfying the exact same sequents. For simplicity of comparison we will just work with partially ordered P-frames here. The significant difference from our approach is that they use only one relation R for defining both $\Diamond$ and $\Box$. In fact our frames, let us call them *D-frames* for now, corresponding to PML are frames $\mathbb{F} = (W, \leq, R_\Diamond, R_\Box)$ where $(W, \leq)$ is a partial order, and $R_\Diamond$ and $R_\Box$ are binary relations on W satisfying

$$\begin{aligned} \leq \circ R_\Diamond \circ \leq &\subseteq R_\Diamond \\ \geq \circ R_\Box \circ \geq &\subseteq R_\Box \end{aligned} \tag{KF}$$

together with the correspondents for the interaction axioms:

$$\begin{aligned} R_\Diamond &\subseteq (R_\Diamond \cap R_\Box) \circ \leq \\ R_\Box &\subseteq (R_\Box \cap R_\Diamond) \circ \geq. \end{aligned} \tag{D}$$

(These can be worked out by a process similar to the one described in Example 4.2.)

To understand how P-frames and D-frames are related we need an observation which will be crucial in comparing our frames to others in other examples as well.

Proposition 6.1. *Let $(W, \leq)$ be a partially ordered set. The operations $\langle R\rangle$, $[R]$, $[R\rangle$, and $\langle R]$ given by a binary relation R on W are operations on $\mathcal{D}(W)$ if and only if*

$$\begin{aligned} \leq \circ R &\subseteq R \circ \leq \\ \geq \circ R &\subseteq R \circ \geq \\ \geq \circ R &\subseteq R \circ \leq \\ \leq \circ R &\subseteq R \circ \geq, \end{aligned} \tag{P}$$

respectively. Furthermore, the largest relations on W for which the operations $\langle R\rangle$, $[R]$,

[1] They actually use the set of up-sets as a universe of the complex algebra. Of course it is equivalent to what is presented here.

$[R\rangle$, *and* $\langle R]$, *respectively, are still on* $\mathcal{D}(W)$ *are:*

$$\leq \circ R \circ \leq$$
$$\geq \circ R \circ \geq$$
$$\geq \circ R \circ \leq$$
$$\leq \circ R \circ \geq.$$

This proposition entails that the relation giving rise to a given DMA$^+$ operation $\Diamond$, $\Box$, $\triangleright$, or $\triangleleft$ on $\mathcal{D}(W)$ is by no means unique. But the ones we are using, which satisfy the conditions (KF), are the largest.

Returning to positive modal logic, notice that the first order correspondents (D) that our D-frames satisfy give us a good hint for which single relation to use in their place, namely $R = R_\Diamond \cap R_\Box$. It is straightforward to check that if $\mathbb{F} = (W, \leq, R_\Diamond, R_\Box)$ is a D-frame, then $\mathbb{G} = (W, \leq, R_\Diamond \cap R_\Box)$ is a P-frame. The original relations can be retrieved as $R_\Diamond = (R_\Diamond \cap R_\Box) \circ \leq \; = \; \leq \circ (R_\Diamond \cap R_\Box) \circ \leq$ and $R_\Box = (R_\Diamond \cap R_\Box) \circ \geq \; = \; \geq \circ (R_\Diamond \cap R_\Box) \circ \geq$. From this it follows that $\mathbb{F}$ and $\mathbb{G}$ have the exact same complex algebras.

As an aside it may be mentioned that this is a special P-frame in the sense that $R = R_\Diamond \cap R_\Box$ satisfies the extra condition

$$(R \circ \leq) \cap (R \circ \geq) \subseteq R. \tag{P$'$}$$

But one can show that for any P-frame $\mathbb{G} = (W, \leq, R)$, the operations $\langle R\rangle$ and $[R]$ are equal to $\langle (R \circ \leq) \cap (R \circ \geq)\rangle$ and $[(R \circ \leq) \cap (R \circ \geq)]$, respectively, and the relation $(R \circ \leq) \cap (R \circ \geq)$ is the largest on W with this property.

Now for the converse, if $\mathbb{G} = (W, \leq, R)$ is a partially ordered P-frame, then $\mathbb{F} = (W, \leq, R \circ \leq, R \circ \geq)$ is a D-frame with the same complex algebra as $\mathbb{G}$.

6.2. DL with an endomorphism

Here we want to consider a situation which may not have so much logical meaning. Nevertheless it will be instructive in making the link between the above example and our later examples. Consider the theory of DLs endowed with an endomorphism, that is, the theory of algebras $\mathbb{A} = (\mathbb{D}, h)$ where $h : A \to A$ is a homomorphism, that is,

$$h(x \wedge y) \approx h(x) \wedge h(y) \text{ and } h(\top) \approx \top \tag{M}$$

$$h(x \vee y) \approx h(x) \vee h(y) \text{ and } h(\bot) \approx \bot. \tag{J}$$

There are several ways to approach this. We could think of h as a $\Diamond$ satisfying the additional identities given in (M), or we could think of h as a $\Box$ satisfying the additional identities given in (J). But from the experience with the positive modal logic example we may realize that it might be interesting to consider this in a slightly different light, namely as an extension of PML given by the addition of

$$\Diamond x \Rightarrow \Box x \text{ and } \Box x \Rightarrow \Diamond x.$$

That is, we may think of it as the theory of DMAs $\mathbb{A} = (\mathbb{D}, \Diamond, \Box)$ satisfying $\Diamond x \approx \Box x$. It is clear that this equation implies the interaction axioms.

In this setting the dual frames would be of the form $\mathbb{F} = (W, \leq, R_\Diamond, R_\Box)$ where $R_\Diamond$ and $R_\Box$ satisfy the condition (KF), together with the formulas

$$\begin{array}{l}\forall u \forall v\, \forall w\, ((u R_{\Diamond} v \wedge u R_{\Box} w) \to v \geq w) \\ \forall u \exists v\ \ (u R_{\Diamond} v \wedge u R_{\Box} v),\end{array} \tag{E}$$

which are the first order correspondents of $\Diamond x \Rightarrow \Box x$ and $\Box x \Rightarrow \Diamond x$, respectively.

Now from the PML setting we know that we can take frames $\mathbb{G} = (W, \leq, R_{\Diamond} \cap R_{\Box})$ as alternate dual frames. Notice that the properties (E) imply

$$\begin{array}{l}\forall u \forall v \forall w\ [(u\,(R_{\Diamond} \cap R_{\Box})\, v \wedge u\,(R_{\Diamond} \cap R_{\Box})\, w) \to v = w] \\ \forall u \exists v\ \ (u\,(R_{\Diamond} \cap R_{\Box})\, v).\end{array} \tag{E$'$}$$

That is, $R_{\Diamond} \cap R_{\Box}$ is a function on W. On the other hand it is simple to show, using $R_{\Diamond} = (R_{\Diamond} \cap R_{\Box}) \circ \leq$ and $R_{\Box} = (R_{\Diamond} \cap R_{\Box}) \circ \geq$, that the conditions (E$'$) imply the conditions (E). Thus alternate semantics for DMAs $\mathbb{A} = (\mathbb{D}, h)$ consisting of a DL and an endomorphism are frames $\mathbb{G} = (W, \leq, R)$ where R satisfies

$$\begin{array}{l}\leq \circ R \ \subseteq\ R \circ \leq \\ \geq \circ R \ \subseteq\ R \circ \geq \\ R \text{ is functional.}\end{array}$$

That is, $R \subseteq W \times W$ is (the graph of) an *order preserving function* on $(W, \leq)$.

6.3. *Negations that reverse join and meet*

Many types of weak negation have been studied, but they generally fall into two main groups: negations that reverse join and meet and negations that are pseudo-complements. Here we will consider the first kind. These fit in well with our previous examples and they include several well-known examples, such as Ockham, MS, De Morgan, Kleene, Stone, and Boolean negations.

6.3.1. *Ockham negations*

The weakest of these are Ockham negations, which just reverse join and meet. That is, algebraically we have the equations:

$$\begin{array}{l} n(x \vee y) \approx n(x) \wedge n(y) \text{ and } n(\bot) \approx \top \\ n(x \wedge y) \approx n(x) \vee n(y) \text{ and } n(\top) \approx \bot. \end{array}$$

That is, we are dealing with algebras consisting of a DL with an anti-endomorphism. In complete analogy to our previous example we may take frames $\mathbb{F} = (W, \leq, R_{\rhd}, R_{\lhd})$ satisfying the condition (KF), together with the formulas

$$\begin{array}{l}\forall u \forall v \forall w\ \ ((u R_{\lhd} v \wedge u R_{\rhd} w) \to v \leq w) \\ \forall u \exists v\ \ (u R_{\rhd} v \wedge u R_{\lhd} v),\end{array} \tag{O}$$

where the properties (O) are the first order correspondents of $\lhd x \Rightarrow \rhd x$ and $\rhd x \Rightarrow \lhd x$, respectively. Now just like in the endomorphism case, we may use semantics based on frames $\mathbb{G} = (W, \leq, R)$ where $R : W \to W$ is an order reversing function on $(W, \leq)$. Also, these alternate frames are obtained from the distributive modal logic frames by setting $R = R_{\rhd} \cap R_{\lhd}$ whereas the distributive modal logic frames are obtained from the alternate frames by setting $R_{\rhd} = R \circ \leq$ and $R_{\lhd} = R \circ \geq$.

6.3.2. MS negations

Now MS negations are Ockham negations satisfying the additional sequent $x \Rightarrow \rhd\rhd x$ whose correspondent is

$$\forall u \exists v\, u R_{\lhd} v R_{\rhd} u,$$

which in the alternate semantics is equivalent to

$$\forall u\; R^2(u) \leq u.$$

6.3.3. De Morgan negations

A De Morgan negation is a negation that is both an MS negation and a dual MS negation and thus it is easy to see that the alternate frames corresponding to De Morgan algebras are frames $\mathbb{G} = (W, \leq, R)$ where $R : W \to W$ is an order reversing involution on $(W, \leq)$.

6.3.4. Kleene negations

A Kleene negation is a De Morgan negation satisfying the additional sequent $x \wedge \rhd x \Rightarrow y \vee \rhd y$ whose correspondent is

$$\forall u (u R_{\rhd} u \vee \forall v (u R_{\rhd} v \to u \leq v)),$$

which in the alternate semantics is equivalent to

$$\forall u (R(u) \leq u \vee u \leq R(u)).$$

6.3.5. Stone negations

A Stone negation is an Ockham negation satisfying the additional sequent $x \wedge \rhd x \Rightarrow \perp$ whose correspondent is

$$\forall u (u R_{\rhd} u),$$

which in the alternate semantics is equivalent to

$$\forall u (R(u) \leq u).$$

This latter property, together with the fact that R is an order reversing map on $(W, \leq)$, easily is shown to imply that R maps each element u of W to the unique minimal element in $(W, \leq)$ below u, thus yielding the well-known characterization of dual spaces of Stone algebras. Notice that this first order correspondent actually restricts the allowable partial orders to those for which each element is above a unique minimal element.

6.3.6. Boolean negations

A Boolean negation is a De Morgan negation which is also a Stone negation. Thus the alternate dual frames are frames $\mathbb{G} = (W, \leq, R)$ where $R : W \to W$ is an order reversing involution on $(W, \leq)$ with $R(u) \leq u$ for all $u \in W$. But this implies that each element of W is equal to the unique minimal element below it, that is, $\leq$ is just equality and R is the identity map. That is, the dual frames are essentially just arbitrary sets. Notice that our dual frames $\mathbb{F} = (W, \leq, R_{\rhd}, R_{\lhd})$ would also be trivial in this case, in the sense that $\leq$ would be equality, and both $R_{\rhd}$ and $R_{\lhd}$ would be the diagonal relation.

Notice that this means that our semantics for classical modal algebras agree with the usual Kripke frames.

7. Conclusions

It will have become obvious to the reader that this paper, although reporting on a coherent piece of research, is in fact part of an ongoing research program, in which numerous natural and interesting generalizations of the present framework remain to be studied or studied further. To mention just two examples: What can be said if we expand our distributive lattices with more general kinds of operations? How many of the results presented here will stand up if we drop the requirement that (logically) the underlying logic is distributive, or (algebraically) the lattice reducts of the algebras are distributive? We refer to Gehrke and Harding [12] for some first explorations concerning the latter question.

But there are, in the context of our distributive modal logic, also a few more specific issues to be addressed.

To start with, the reader will have observed the striking difference in proof strategies: correspondence was proved by a reduction to the classical (Boolean) case, canonicity was not. This difference in approach begs a number of questions.

For instance, why did we not try to reduce the canonicity proof for Sahlqvist DML sequents to the classical case as well? We had two reasons for not doing so. First, in our view, our canonicity proof for distributive modal logics can be seen as one specific outcome of a theory on canonical extensions, and the classical Sahlqvist canonicity as another such outcome; we see no reason to believe that the case of classical modal logic (or, algebraically, Boolean algebras with operators) occupies a more fundamental position in this theory; cf. also our discussion in the previous section. And second, a reduction to the classical result for canonicity seems to be much *harder* than for correspondence, due to the following reason. In the correspondence case, where we are working with perfect DMAs, there is an obvious way to connect with Boolean algebras with operators, namely by taking the (Boolean) complex algebra of the dual *frame*. In the canonicity case however, we would need to embed arbitrary DMAs into BAOs in a way that would interact nicely with taking canonical extensions, and we do not see a natural, general way for doing so.[2]

On the other hand, we do believe that the correspondence proof could be treated much more algebraically, and also, without a reduction to the classical case. In fact such a treatment, in the case of DMLs whose basic operations are both operators and dual operators (that may have any arity), is the subject of ongoing research by one of the authors. This approach may be particularly advantageous later when it comes to the treatment of non-distributive lattice expansions.

Notice that these questions may be interesting in their own right, but also connect to a line of recent work addressing the question whether classical modal logic with one single diamond can 'simulate all others'; cf. [17,23]. For, the translations B_ε of Section 4 can easily be extended to the level of *logics*, so as to connect distributive modal logics to classical modal logics (in the language with $\Diamond_\leq$, $\Diamond_\geq$, $\Diamond_\Diamond$, $\Diamond_\Box$, $\Diamond_\triangleright$ and $\Diamond_\triangleleft$). One may wonder whether there is always an order type ε such that B_ε induces a *simulation* of logics, and if so, whether this simulation would be nice in preserving and/or reflecting interesting properties of logics.

[2] Using a Galois connection one can interpret a DMA in a BAO, see for instance Harding [19], but this generally destroys all readily available canonicity results.

Second, the reader may have wondered why there is no mention of *intuitionistic* modal logic (or Heyting algebra based modal algebras) in our paper. It is definitely possible to handle intuitionistic modal logic in our framework, but we have not gone into details here as this topic fits more naturally in a setting where one allows additional operations of higher arity than one. Observe that the intuitionistic implication takes meets in the first, and joins in the second coordinate, to meets in the codomain; this means that Heyting implication is a binary dual operator (with a flip in the second coordinate). Handling such (dual) operators of higher arity is certainly possible, but a bit more care must be taken, and fewer formulas will be 'Sahlqvist' because these basic operations may be *non-smooth*; that is, their σ- and their π-extensions may not agree. In fact, one may show, cf. [15], that this is exactly what happens for the implication in most infinite Heyting algebras, and the π-extension of the Heyting implication is the one that makes the canonical extension into a Heyting algebra. To give some indication where smoothness comes in, notice that smoothness was used in proving that left and right Sahlqvist terms are stable; cf. Lemma 5.10. Nevertheless, a lot can be said and done in this case, as witnessed already by Ghilardi and Meloni [16].

Since there are a lot of similarities between the latter paper and ours, the connection between our work and that of Ghilardi and Meloni is the third and last issue that we will address. These similarities go much further than that canonicity is treated independently from correspondence; in particular, the work of Ghilardi and Meloni also crucially involves the extension of monotone lattice maps to maps between the canonical extension. (Note, however, that their definition uses, as an intermediate level, partial maps on the collection of filters and ideals of the original lattice; the connection with our approach lies in the correspondence between filters and ideals on the one hand, and closed and open elements, on the other.) A seemingly big difference is that these authors work constructively; however, we believe that our results could most likely be reworked not to depend on the axiom of choice. Apart from this issue, it is certainly possible to make a more precise comparison between their proof methods and ours; for instance, we believe that the main results of [16] could be reformulated in our terminology and proved using our methods. (Of course, since Ghilardi and Meloni work with a larger set of operations, including intuitionistic implication, this would involve an extension of our work as indicated above.) For reasons of space limitations, we refrain from doing so here.

Ghilardi and Meloni show a large class of formulas to be canonical: when it comes to intuitionistic modal logic, their results are probably the most general available. Restricting to our language of distributive modal logic, we would like to compare the scope of their results to that of ours, but this seems to be rather difficult. The authors of [16] do not make any explicit claims concerning our language, and it is not straightforward to extract results concerning our language from their work. In any case, we believe that the issue of canonicity is of sufficient interest that various proof methods could, and in fact should, exist side by side.

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